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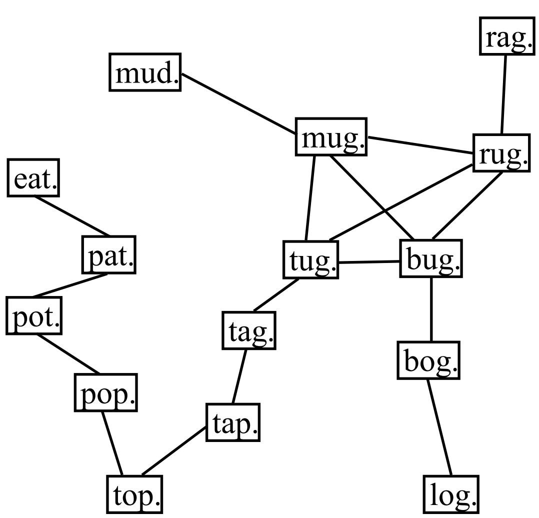
50 YRS
1963-2013
Enriching Mathematics and
Computer Science for 50 years

Grade 7 & 8 Math Circles
Graph Theory Solutions

1.
   - Two vertices labelled as 'B'.
   - An edge 'loops' from B a vertex to the same vertex.
   - Two edges go between the same pair of vertices.
   - An edge doesn't end a vertex.

2.



3. (a) g, i, j, e, b, f, h, c, a, d, g

   (b) i, viii, v, x, vi, ix, xii, iii, ii, vii, iv, xi, i

4. Mathtopia, Circle City, Fraction Falls, Radius River, Times Town, Equationville, Gausston, Vertex Village, Mathtopia

5. (a) 2. We could change the first digit of one number to match the first digit of the second number. Then repeat with the second digit. This results in a path of length 2. If one digit is already equivalent we only need to change one digit, resulting in a path of length 1. We can take a path between any two numbers in this way, meaning there is a path between any two vertices of length 2 or 1.

   (b) 5. Similar reasoning to *part a*.

   (c) 142 874; 942 874; 982 874; 985 874; 985 374; 985 304

   (d) degree 139 = 26. All adjacent vertices are: 239; 339; 439; 539; 639; 739; 839; 939; 109; 119; 129; 149; 159; 169; 179; 189; 199; 130; 131; 132; 133; 134; 135; 136; 137; and 138.

6. (a) i. \$300

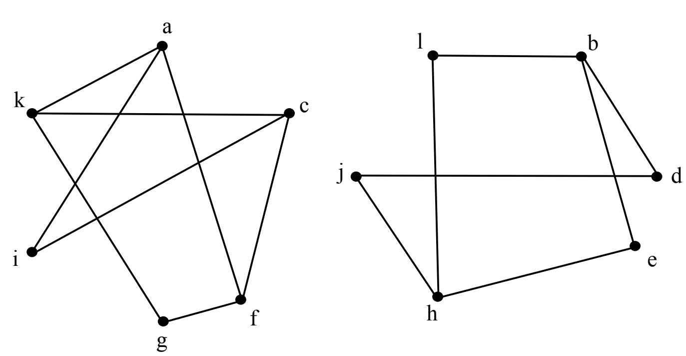
      ii. Brazil. The degree of Brazil is 10; much greater than any of the others.

      iii. \$400 Venezuela to Paraguay; \$500 French Guiana to Bolivia.

      iv. Chile, Peru, Columbia, Venezuela, Guyana, Suriname; \$400.

   (b) Peru, Ecuador, Columbia, Venezuela, Guyana, Suriname, French Guiana, Brazil, Paraguay, Bolivia, Chile, Argentina, Uruguay.

7. (a) If you redraw this graph we can get:



   This shows that the graph is not connected. So there cannot be a path from c to j, which means no cycle can contain both vertices c and j. Thus no cycle can contain all the vertices, and so a Hamilton cycle does not exist in this graph.

(b) Observe vertices 5 and 6. Both have a degree of 1. All vertices in a cycle must have a degree of 2. One edge must connect the vertex to the cycle, and the second to 'leave' the vertex. Since 5 and 6 only have a degree of 1, there is no way to 'leave' either vertex once connected to the cycle's path. So vertices 5 and 6 cannot be in any cycle, so a cycle containing all vertices does not exist.

8. (a) Not isomorphic. In the first graph, there are two cycles of length 3. In the second graph, there are no cycles of length three. If these graphs were isomorphic then there should be the same number and length of cycles in both graphs. This is not the case so these graphs are not isomorphic.

   (b) Isomorphic.

$f(1)$=e	$f(2)$=a	$f(3)$=b	$f(4)$=f
$f(5)$=g	$f(6)$=c	$f(7)$=d	$f(8)$=h

   (c) Isomorphic.

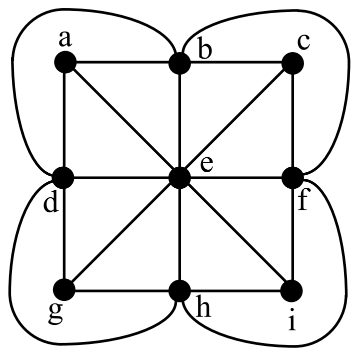
$f(1)$=a	$f(2)$=i	$f(3)$=h	$f(4)$=c	$f(5)$=b
$f(6)$=j	$f(7)$=g	$f(8)$=f	$f(9)$=e	$f(10)$=d

9. Answers will vary. See lesson for an example.

10. (a) Let's begin with a simple tree; 2 vertices with 1 edge connecting them. All ways you can draw this graph are isomorphic to one another. Now, add one vertex. Since we want this to be a tree, we must also add an edge so the graph is connected. Now if we try to add another edge, there will be a cycle in the graph. So there must be 3 vertices and 2 edges in this tree. If we keep repeating this process, we will always add one vertex and one edge. Trying to add more than one edge will result in a cycle, and so we won't have a tree. Thus there will always be one less edge in a tree than there are vertices. So if there are V vertices, then there are V-1 edges.

(b) In any graph, one edge will connect two vertices. Thus each edge in a graph will contribute two degrees to the graph. So for any graph, the sum of degrees of all vertices will be twice the number of edges. We have 20 vertices and each has degree 3. So $20 \times 3 = 60$ which is the sum of degrees of all vertices. This value equals twice the number of edges. So $60 \div 2 = 30$. Thus there are 30 edges in this graph.

11. (a)



(b)