Math Circles: Graph Theory III

Centre for Education in
Mathematics and Computing

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1 Notation

Consider a Rubik's cube, as shown in Figure 1. The letters U, F, R, L, B, and D shall refer respectively to clockwise 90° turns of the upper, front, right, left, back, and downward faces of the cube, as shown in Figure 1. The corresponding counterclockwise 90° turns are denoted U^{-1}, F^{-1}, R^{-1}, L^{-1}, B^{-1}, and D^{-1}, and 180° turns are denoted U^2, F^2, R^2, L^2, B^2, and D^2.

Sequences of moves are read from left to right, so for example LFU means: "Turn the left face 90° clockwise, then turn the front face 90° clockwise, and then turn the upper face 90° clockwise. Observe that $FF = F^2$ and $FFF = F^{-1}$. A sequence of moves is called an *operation*.

Exponents denote repeated moves, as in $F^3 = FFF = F^{-1}$ and $(R^{-1}U)^3 = R^{-1}UR^{-1}UR^{-1}U$. The trivial sequence of moves (i.e. no operation at all) is denoted $\mathbb{1}$, so for example $F^4 = \mathbb{1}$.

The x-axis refers to the axis passing through the centers of the left and right faces of the cube. Similarly, the y-axis refers to the axis passing through the centers of the front and back faces, and the z-axis refers to the axis passing through the centers of the upper and downward faces.

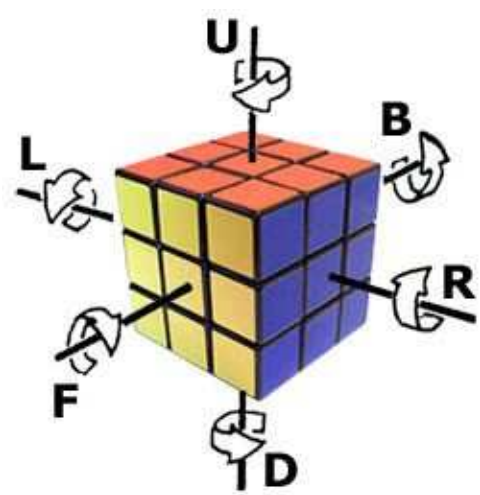
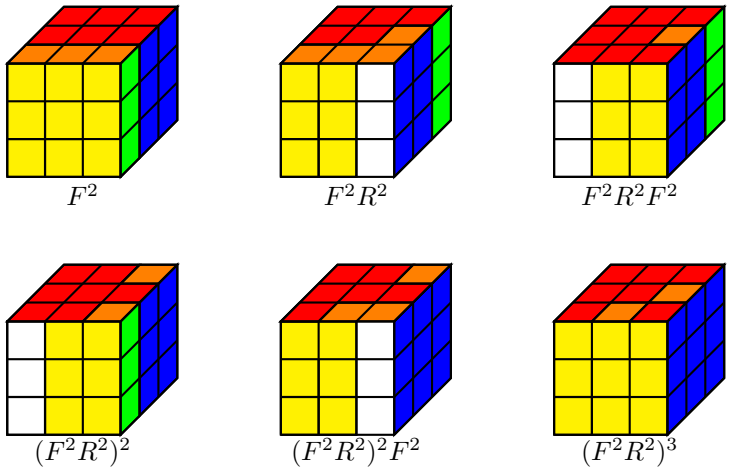


Figure 1: Notation for moves on a Rubik's cube.

2 Cayley graphs

Orient the cube as in Figure 1, so that the red face is on top and the yellow face is in the front. Using only the operations F^2 and R^2, and combinations thereof, what cube positions can you attain? For example, if we alternate F^2 and R^2, we obtain the following sequence of positions:



If we graph all the possible positions and use edges to denote which moves are used to navigate between which positions, we obtain the graph shown in Figure 2.

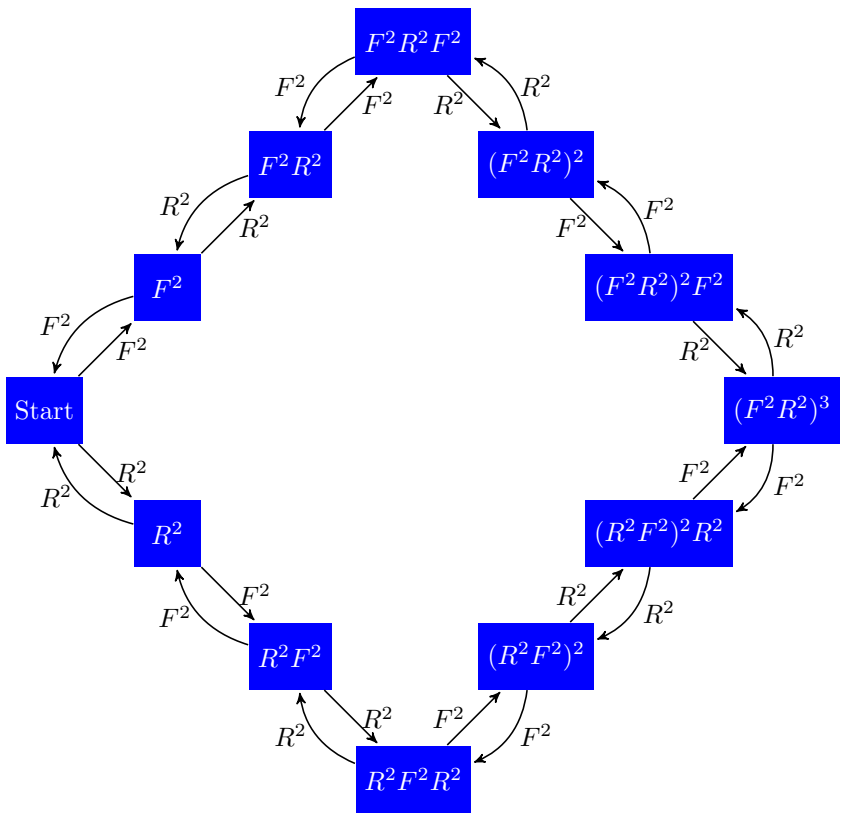
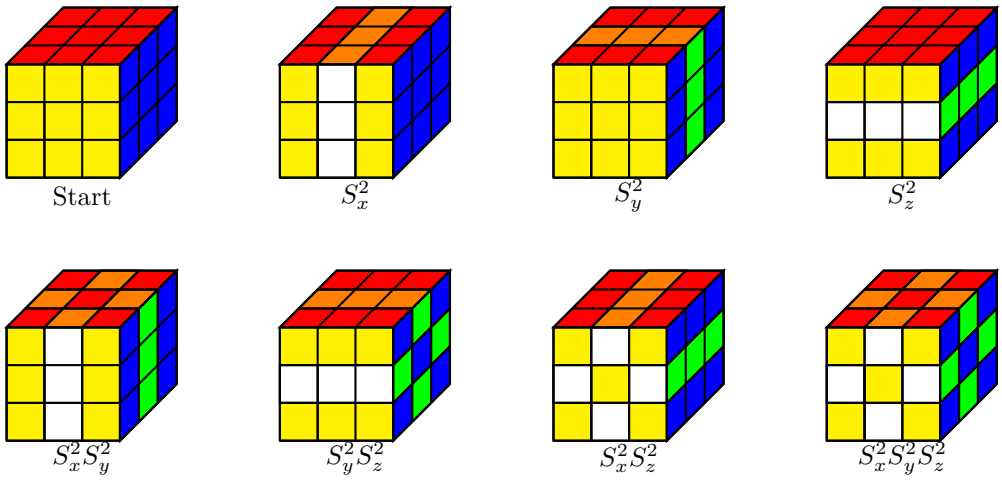


Figure 2: Cayley graph generated by $\{F^2, R^2\}$

A graph of this form is called a *Cayley graph*. Such a graph depicts some subset of positions on a Rubik's cube, along with the moves that can be used to transform one position into another position. In the example of Figure 2, we say the graph is *generated by* the elements $\{F^2, R^2\}$. This phrase means that we consider only the positions attainable via some sequence of these two moves, and we depict only those edges corresponding to exactly one or the other of these moves. For example, even though we could get from the F^2 position

to the $F^2R^2F^2$ position via the compound move R^2F^2, we do not include any edge labeled R^2F^2 going between these two positions, because R^2F^2 is not in the set $\{F^2, R^2\}$.

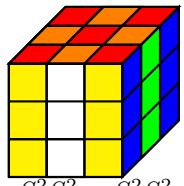
For another example, let S_x (respectively S_y and S_z) denote the "slice" move obtained by rotating the middle layer of the cube clockwise 90° around the x-axis (respectively, the y-axis and z-axis). Here the clockwise direction is determined by looking from the positive direction of the axis. Consider the Cayley graph generated by $\{S_x^2, S_y^2, S_z^2\}$. We obtain the following eight positions:



One can check that these are the only eight positions attainable using combinations of S_x^2, S_y^2, and S_z^2. The resulting Cayley graph is depicted in Figure 3. One big difference between this Cayley graph and the Cayley graph generated by $\{F^2, R^2\}$ is that the generators $\{S_x^2, S_y^2, S_z^2\}$ *commute*, meaning that the resulting cube position does not depend on the order in which the operations are performed. For example,

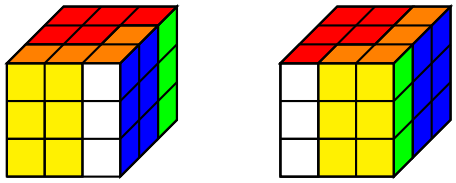
$$S_x^2S_y^2 = S_y^2S_x^2$$

in the sense that both $S_x^2S_y^2$ and $S_y^2S_x^2$ yield the same cube position, namely:



$S_x^2S_y^2 = S_y^2S_x^2$

In contrast, the operations F^2 and R^2 do not commute, since F^2R^2 does not lead to the same cube position as R^2F^2:



F^2R^2 $\qquad$ R^2F^2

In general, commuting operators lead to Cayley graphs with fewer vertices and more edges than non-commuting operators, since in the case where operators commute, different orderings of the operators lead to the same positions, whereas for non-commuting operators, the resulting positions depend on the order in which the operators are applied.

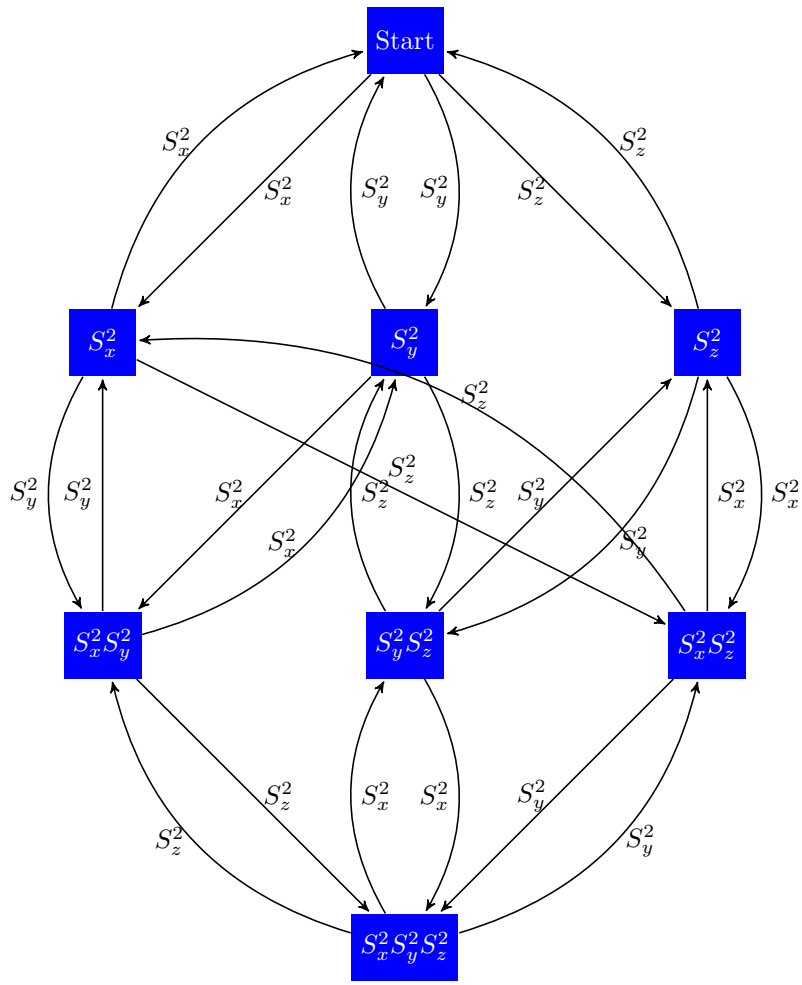


Figure 3: Cayley graph generated by the three "slice" moves $\{S_x^2, S_y^2, S_z^2\}$

3 Commutators

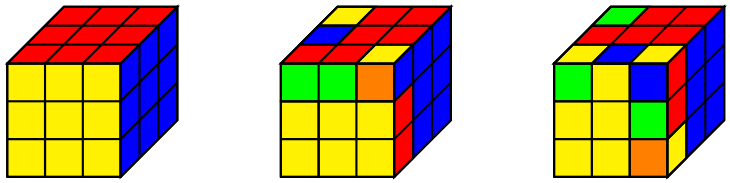
In ordinary arithmetic, the commutative law always holds: $xy = yx$ for all numbers x and y. On a Rubik's cube, the commutative law does not always hold, as we have seen. We now study the properties of non-commutative operations such as those of the Rubik's cube.

Let g and h denote any two sequences of operations on the cube. Then g and h commute if and only if $gh = hg$. If we let g^{-1} (respectively h^{-1}) denote the *inverses* of g and h (that is, the reverse sequence of moves of g, done in reverse order, so that $gg^{-1} = g^{-1}g = hh^{-1} = h^{-1}h = \mathbb{1}$), then we have

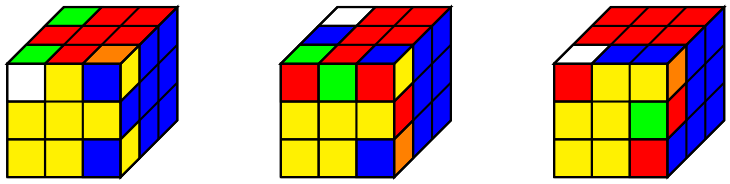
$$gh = hg \iff h^{-1}gh = h^{-1}hg \iff h^{-1}gh = g \iff g^{-1}h^{-1}gh = g^{-1}g \iff g^{-1}h^{-1}gh = \mathbb{1}.$$

The compound sequence $g^{-1}h^{-1}gh$ is called the *commutator* of g and h, and its significance lies in the fact that g and h commute if and only if their commutator is $\mathbb{1}$. In the case where they do not commute, the position produced by the commutator indicates exactly how much the commutative property fails.

Example 3.1. Let $g = U$ and $h = F$. Then the commutator $g^{-1}h^{-1}gh = U^{-1}F^{-1}UF$ generates the following six positions:



Start $\quad U^{-1}F^{-1}UF \quad (U^{-1}F^{-1}UF)^2$



$(U^{-1}F^{-1}UF)^3 \quad (U^{-1}F^{-1}UF)^4 \quad (U^{-1}F^{-1}UF)^5$

If we continue from $(U^{-1}F^{-1}UF)^5$, we find that we return to the starting position: $(U^{-1}F^{-1}UF)^6 = \mathbb{1}$. In such a situation we say that $U^{-1}F^{-1}UF$ has *order* equal to 6, since $(U^{-1}F^{-1}UF)^6 = \mathbb{1}$ and no smaller power of $U^{-1}F^{-1}UF$ is equal to $\mathbb{1}$.

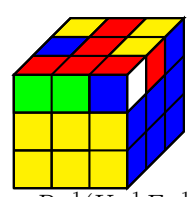
4 Conjugation

For any two operations g and h, the *conjugate* or *conjugation* of h by g is the operation $g^{-1}hg$. Observe that

$$hg = gh \iff g^{-1}hg = g^{-1}gh \iff g^{-1}hg = h.$$

Hence the conjugation of h by g is equal to h if and only if g and h commute. If g and h do not commute, then the conjugate (similar to the commutator) tells us how badly they fail to commute.

Example 4.1. Let $h = U^{-1}F^{-1}UF$ be the commutator from Example 3.1. We conjugate h by $g = R$ to obtain:



$g^{-1}hg = R^{-1}(U^{-1}F^{-1}UF)R$

Since h has order 6, we observe that the conjugate $g^{-1}hg$ must also have order 6:

$$\begin{aligned}(g^{-1}hg)^6 &= (g^{-1}hg)(g^{-1}hg)(g^{-1}hg)(g^{-1}hg)(g^{-1}hg)(g^{-1}hg)\\ &= g^{-1}h(gg^{-1})h(gg^{-1})h(gg^{-1})h(gg^{-1})h(gg^{-1})hg\\ &= g^{-1}h^6g = g^{-1}\mathbb{1}g = \mathbb{1}.\end{aligned}$$

The operation $g^{-1}hg = R^{-1}(U^{-1}F^{-1}UF)R$ has the interesting property that it leaves the bottom two-thirds of the cube intact. We will exploit this property in the next section, where we use this operation to obtain a procedure to solve the cube.

5 Solving the cube

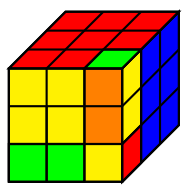
Commutators and conjugation are the building blocks used to solve the cube. In fact, one can solve the cube using nothing more than these operations alone. The resulting method is somewhat slow and inefficient compared to the fastest known methods, but it has the advantage of being very simple from a theoretical viewpoint.

We first introduce one additional piece of terminology. By a *cubie* or *piece*, we mean a single indivisible 1×1 mini-cube within the 3×3 cube. A 3×3 Rubik's cube has eight corner cubies (having three stickers each), twelve edge cubies (having two stickers each), six center cubies (having one sticker each), and one central cubie in the very center of the cube.

The *position* of a cubie denotes its spatial position (ignoring orientation) relative to the six center cubies. Note that the six center cubies never change position. The *orientation* of a corner cubie denotes the angle by which it is rotated relative to its correct alignment ($0°, 120°$, or $240°$). Similarly, the *orientation* of an edge cubie denotes the angle by which it is rotated relative to its correct alignment ($0°$ or $180°$).

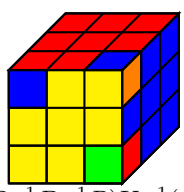
5.1 Positioning corner pieces

Consider the conjugation of D by R, given by $R^{-1}DR$, as shown here:



$R^{-1}DR$

Notice that this operation affects only one single cubie on the top layer of the cube. Hence, if we take the commutator of $g = R^{-1}DR$ and, say, $h = U$, we obtain a cube position in which only the top front right, top front left, and bottom front right corner cubies are affected:



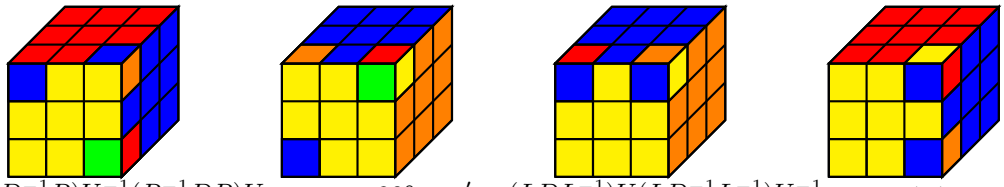
$g^{-1}h^{-1}gh = (R^{-1}D^{-1}R)U^{-1}(R^{-1}DR)U$

Denote $g^{-1}h^{-1}gh$ by r. Then the operation r cycles exactly three corner cubies, and affects no other cubies. By repeated application of r, it is possible to place all the corner cubies into their proper positions, although their orientations will most likely be wrong.

5.2 Orienting corner pieces

For any operation g, the *mirror image* of g, denoted g', is the sequence of moves that one obtains by performing the operation g and reflecting the entire cube and operation in a mirror. For example, if $r = (R^{-1}D^{-1}R)U^{-1}(R^{-1}DR)U$, then $r' = (LDL^{-1})U(LD^{-1}L^{-1})U^{-1}$.

As we have seen, the operation r cycles exactly three corner cubies, and affects no other cubies. If we repeat r several times, we find that $r^3 = 1$, so that r has order 3. However, if we apply r and then *rotate* the entire cube clockwise along the y-axis, we can then apply the mirror image r' of r, which will also undo the cycling, and restore all the cubies to their original positions. Crucially, although this combination restores all the cubies to their original positions, it does *not* restore all the cubies to their original orientations; two of the cubies are mis-oriented:



$r = (R^{-1}D^{-1}R)U^{-1}(R^{-1}DR)U$ rotate 90° $r' = (LDL^{-1})U(LD^{-1}L^{-1})U^{-1}$ un-rotate

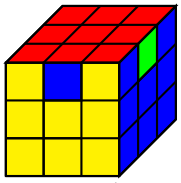
If we rewrite this operation to eliminate the rotation, we obtain the *twist* operation

$$t = (R^{-1}D^{-1}R)U^{-1}(R^{-1}DR)U(ULU^{-1})R(UL^{-1}U^{-1})R^{-1},$$

which rotates the top front right corner cubie by 120°, the bottom front right corner cubie by 240°, and leaves all other cubies unaffected. By repeated application of the twist operation, it is possible to orient all the corner cubies correctly.

5.3 Positioning edge pieces

Consider the conjugate $m = R^{-1}U^{-1}F^{-1}UFR$ from Example 4.1. The inverse of m is $m^{-1} = R^{-1}F^{-1}U^{-1}FUR$. The mirror image of m is $m' = LUFU^{-1}F^{-1}L^{-1}$, and the inverse of the mirror image is $m'^{-1} = LFUF^{-1}U^{-1}L^{-1}$. By combining these operations, it is possible to manipulate the edge pieces without affecting any of the corner pieces. For example, the operation $mm'^{-1} = (R^{-1}U^{-1}F^{-1}UFR)(LFUF^{-1}U^{-1}L^{-1})$ cycles exactly three edge cubies, and leaves all other cubies unaffected:

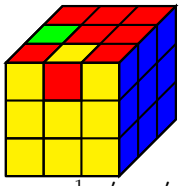


mm'^{-1}

Using this operation, it is possible to place all the edge cubies into their proper positions (possibly with wrong orientations).

5.4 Orienting edge pieces

If we conjugate m and m'^{-1}, we obtain the *edge flip* operation $f = m^{-1}m'mm'^{-1}$, which reverses the orientation of two adjacent edge cubies while leaving all other cubies untouched:



$f = m^{-1}m'mm'^{-1}$

Using this operation, it is possible to orient all the edge cubies correctly, and thereby solve the cube.