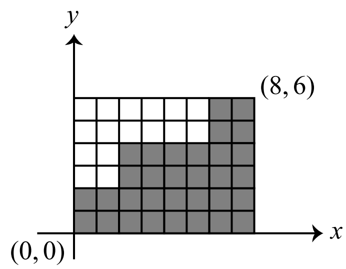
Intermediate Math Circles
November 27, 2013
CIMC and My favorite problems

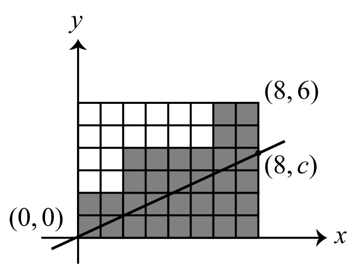
CIMC A4

An 8×6 grid is placed in the first quadrant with its edges along the axes, as shown. A total of 32 of the squares in the grid are shaded. A line is drawn through $(0, 0)$ and $(8, c)$ cutting the shaded region into two equal areas. What is the value of c?



Solution 1

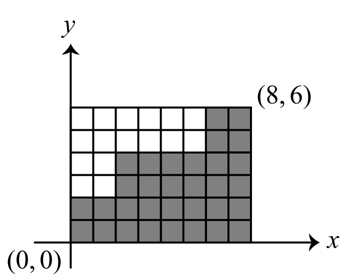
Since 32 of the 1×1 squares are shaded, then the total shaded area is 32.
We want to draw a line through $(0, 0)$ and $(8, c)$ that divides the shaded region into two pieces, each with area $\frac{32}{2} = 16$.
As long as the slope of the line segment through $(0, 0)$ and $(8, c)$ is not too large, the bottom piece will be a triangle with vertices $(0, 0)$, $(8, 0)$, and $(8, c)$.

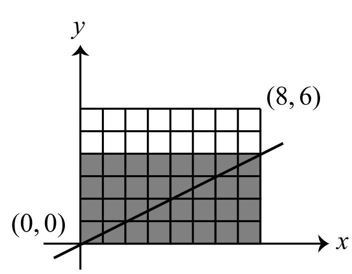


This triangle is right-angled at $(8, 0)$ and so has base of length 8 and height c.
For the area of this triangle to be 16, we need $\frac{1}{2}(8)(c) = 16$ or $4c = 16$.
Thus, $c = 4$.
(Note that if $c = 4$, the line segment passes through $(0, 0)$ and $(8, 4)$. This line segment has slope $\frac{4-0}{8-0} = \frac{1}{2}$ and so passes through the points $(2, 1)$ and $(4, 2)$. This confirms that the line segment does not pass through any unshaded regions.)

Solution 2

We move the upper right shaded 2×2 square to complete a 8×4 shaded rectangle, as shown.





The shaded area is now a rectangle with bottom left vertex $(0, 0)$ and top right vertex $(8, 4)$.
The area of the rectangle is cut in half by its diagonal passing through $(0, 0)$ and $(8, 4)$.
Note the region affected by this move lies above this diagonal. (This is because the diagonal has

slope $\frac{4-0}{8-0} = \frac{1}{2}$ and so passes through the points $(2,1)$ and $(4,2)$.)
This means that the areas of the regions below and above the line were unaffected by the move.
Thus, reversing the move does not change the areas of the regions below and above the line.
Therefore, the line drawn through $(0,0)$ and $(8,4)$ cuts the original region into two pieces of equal area.
Thus, $c = 4$.

CIMC A5

A *palindrome* is a positive integer that is the same when read forwards or backwards. For example, 1331 is a palindrome. Determine the number of palindromes between 1000 and 10 000 that are multiples of 6.

SOLUTION

Since 1000 and 10 000 are not palindromes, then every palindrome between 1000 and 10 000 has four digits and so has the form $ABBA$ for some digits A and B with $A \neq 0$.
We are looking for palindromes $ABBA$ that are divisible by 6.
An integer is divisible by 6 whenever it is divisible by 2 (that is, it is even) and divisible by 3.
Since we want $ABBA$ to be even, then A must be even.
Therefore, $A = 2$, $A = 4$, $A = 6$, or $A = 8$.
For a positive integer to be divisible by 3, the sum of its digits must be divisible by 3.
We look at each possible value for A separately and determine values of B that give palindromes that are divisible by 6:

- Case 1: $A = 2$
  Here, $ABBA = 2BB2$ and the sum of its digits is $2 + B + B + 2 = 2B + 4$.
  We need to determine all digits B for which $2B + 4$ is divisible by 3.
  Since B is at most 9, then $2B + 4$ is at most $2(9) + 4 = 22$. Also, $2B + 4$ is even because $2B + 4 = 2(B + 2)$.
  Thus, $2B + 4$ must be an even multiple of 3 that is at most 22.
  Therefore, $2B + 4$ could equal 6, 12 or 18.
  If $2B + 4 = 6$, then $2B = 2$ or $B = 1$.
  If $2B + 4 = 12$, then $2B = 8$ or $B = 4$.
  If $2B + 4 = 18$, then $2B = 14$ or $B = 7$.
  (Alternatively, we could have checked each of the values of B from 0 to 9 individually by dividing $2BB2$ by 6.)

- Case 2: $A = 4$
  Here, $ABBA = 4BB4$.
  Using a similar method to that in Case 1, we determine that the values of B for which $4BB4$ is divisible by 6 are $B = 2, 5, 8$.

- Case 3: $A = 6$
  Here, $ABBA = 6BB6$.
  Using a similar method to that in Case 1, we determine that the values of B for which $6BB6$ is divisible by 6 are $B = 0, 3, 6, 9$.

- Case 4: $A = 8$
  Here, $ABBA = 8BB8$.
  Using a similar method to that in Case 1, we determine that the values of B for which $8BB8$ is divisible by 6 are $B = 1, 4, 7$.

Therefore, there are $3 + 3 + 4 + 3 = 13$ palindromes between 1000 and 10 000 that are multiples of 6.

CIMC A6

Note that $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$ and $\frac{1}{6}-\frac{1}{7}=\frac{1}{6\times 7}$ and

$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{42}=1.$

Determine one triple (x,y,z) of positive integers with $1000<x<y<z<2000$ and

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{45}=1$$

SOLUTION

The given fact that $\frac{1}{6}-\frac{1}{7}=\frac{1}{6\times 7}$ suggests that we consider $\frac{1}{n}-\frac{1}{n+1}$.

Using a common denominator,

$$\frac{1}{n}-\frac{1}{n+1}=\frac{n+1}{n(n+1)}-\frac{n}{n(n+1)}=\frac{(n+1)-n}{n(n+1)}=\frac{1}{n(n+1)} \qquad (*)$$

Starting with

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{42}=1$$

we subtract and add the same quantities from the left side to obtain

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{42}-\frac{1}{43}+\frac{1}{43}-\frac{1}{44}+\frac{1}{44}-\frac{1}{45}+\frac{1}{45}=1$$

We then regroup the terms on the left side as

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\left(\frac{1}{42}-\frac{1}{43}\right)+\left(\frac{1}{43}-\frac{1}{44}\right)+\left(\frac{1}{44}-\frac{1}{45}\right)+\frac{1}{45}=1$$

and use the fact $(*)$ above to obtain

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{42\times 43}+\frac{1}{43\times 44}+\frac{1}{44\times 45}+\frac{1}{45}=1$$

This gives

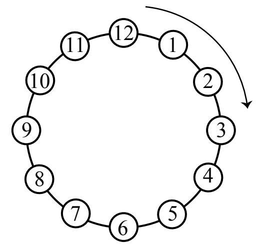
$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{1806}+\frac{1}{1892}+\frac{1}{1980}+\frac{1}{45}=1$$

Therefore, a solution to the given equation that satisfies the condition $1000<x<y<z<2000$ is $(x,y,z)=(1806,1892,1980)$.

There are other solutions to this equation obeying the given conditions, but we are only asked to find one such solution.

CIMC B2

In each part of this problem, cups are arranged in a circle and numbered $1, 2, 3$, and so on. A ball is placed in cup 1. Then, moving clockwise around the circle, a ball is placed in every nth cup. The process ends when cup 1 contains two balls. For example, starting with 12 cups and placing a ball in every 3rd cup, balls are placed, in order, in cups 1, 4, 7, 10, 1.



a) There are 12 cups in the circle and a ball is placed in every 5th cup, beginning and ending with cup 1. List, in order, the cups in which the balls are placed.

b) There are 9 cups in the circle and a ball is placed in every 6th cup, beginning and ending with cup 1. List the numbers of the cups that do not receive a ball.

c) There are 120 cups in the circle and a ball is placed in every 3rd cup, beginning and ending with cup 1. How many cups *do not* contain at least one ball when the process is complete? Explain how you obtained your answer.

d) There are 1000 cups in the circle and a ball is placed in every 7th cup, beginning and ending with cup 1. Determine the number of the cup into which the 338th ball is placed.

SOLUTION

a) The 1st ball goes in cup 1.
Each successive ball goes 5 cups further along the sequence. We obtain the cup number for the next ball by adding 5 to the current cup number. If the total is no larger than 12, this is the cup number of the next ball; if the cup number is larger than 12, we subtract 12 to obtain the actual cup number.
Thus, the 2nd ball goes in cup $1 + 5 = 6$, the 3rd ball goes in cup $6 + 5 = 11$, and the 4th ball goes in cup "$11 + 5 = 16$" which is $16 - 12 = 4$ cups beyond cup 12, so is really cup 4.
We continue until we place a second ball in cup 1.
In order, the balls are placed in cups 1, 6, 11, 4, 9, 2, 7, 12, 5, 10, 3, 8, 1.

b) The first ball goes in cup 1.
The next ball goes in cup $1 + 6 = 7$.
The next ball goes in cup $7 + 6 = 13$, which is the same as cup $13 - 9 = 4$.
The next ball goes in cup $4 + 6 = 10$, which is the same as cup $10 - 9 = 1$.
This means that the process stops.
Therefore, balls are placed in 3 cups in total (cups 1, 7, 4) and so cups 2, 3, 5, 6, 8, and 9 do not receive a ball.

c) The first ball goes in cup 1.
Cup 1 will be reached again after moving 120 cups around the circle.
Since a ball is placed in every 3rd cup, then 120 cups corresponds to $\frac{120}{3} = 40$ additional balls, and the 41st ball will be placed in cup 1 and the process stops.
Therefore, there will be 40 cups in which a ball is placed. (Cup 1 will receive two balls.)
Since 40 cups receive a ball, then $120 - 40 = 80$ cups do not contain at least one ball.

d) The first ball goes in cup 1.
Since each ball is placed 7 cups further along the circle, then the 338th ball will be placed $337 \times 7 = 2359$ cups further along the circle from cup 1.
Cup 1 itself is 1000 cups further along the circle, so we can think of moving 1000 cups to cup 1, then another 1000 cups to cup 1 again, and then 359 further cups to cup 360.
Therefore, the 338th ball is placed in cup 360.

Favorite Question 1

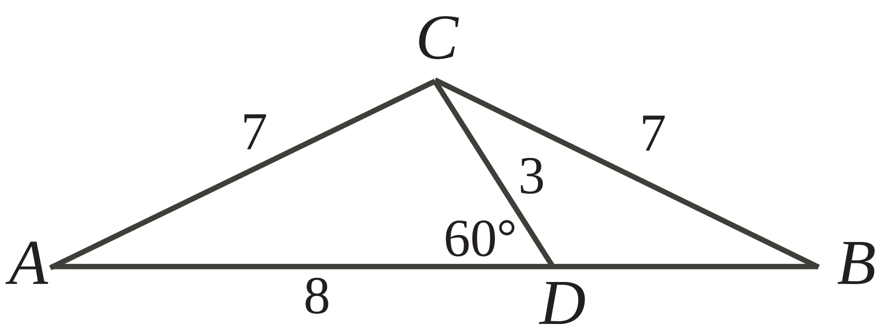
There is a very famous hockey number $9^{99} - 4$. In the expansion of this number "n" is the number of digits, "f" is the first digit, and "l" is the last digit.
Find $n + f + l$.

SOLUTION

From my calculator $9^{99} - 4 \doteq 2.951 \times 10^{94}$
This tells me that the first digit is 2 (second is 9, third is 5, etc.) and there are 95 digits when the decimal place is moved 94 positions right.
Therefore $f = 2$ and $n = 95$.
Look at powers of 9. $9^1 = 9, 9^2 = 81, 9^3 = 729, 9^4 = ...1$, etc.
Since the exponent 99 is odd we know that 9^{99} will end in a 9 and then $9^{99} - 4$ will end in a 5.
Therefore $l = 5$ giving $n + f + l = 95 + 2 + 5 = 102$

Favorite Question 2

In the diagram, $\triangle ABC$ is isosceles with $AC = BC = 7$. Point D is on AB with $\angle CDA = 60°$, $AD = 8$, and $CD = 3$. Determine the length of BD.



SOLUTIONS
An ugly solution can be done using Sine Law and Cosine Law on the non right angle triangles in the diagram.
A nicer solution can be done by drawing a perpendicular line from AB to C meeting AB at E. Then $\triangle CED$ is a $30°60°90°$ triangle with base $ED = 1.5$. Then $AE = EB = 6.5$ and leaves $DB = 5$.
The best solution is to draw a line from C to AD meeting AD at F such that $\angle CFD = 60°$. This gives an equilateral triangle with $FD = 3$ leaving $AF = 5 = DB$ by symmetry.

Favorite Question 3

The Compulsive Ant never turns left, only turns right and always walks one centimeter further after each turn.
Our nutty ant comes out of its house (hill) and walks 1cm East, turns right and walks 2cm South, turns right and walks 3cm West, turns right and walks 4cm North, turns right and walks 5cm East, and so on and so on.

After 2013 of these walks the ant stops for a break.

a) How far has the ant walked in total?

b) How far is the ant from the house?

SOLUTIONS

a) Total Distance$= 1 + 2 + 3 + \cdots + 2012 + 2013 = \frac{2013\times 2014}{2} = 2027091$.

b) Let the ant's home be at the origin and define East as right and North as up.
Then the first 10 moves leave the ant at
$(1, 0), (1, -2), (-2, -2), (-2, 2).(3, 2), (3, -4), (-4, -4), (-4, 4), (5, 4), (5, -6)$.
The x-coordinates can be shown to follow the pattern 1,1,-2,-2,3,3,-4,-4,5,5,.... For an even number of moves "n" the magnitude of the x-coordinate is $\frac{n}{2}$. For an odd number of moves "n" the magnitude of the x-coordinate is $\frac{n+1}{2}$. If the magnitude is odd the x-coordinate is positive and if the magnitude is even then the x-coordinate is negative.
Thus for $n = 2013$ the magnitude is $\frac{2013+1}{2} = 1007$ which is odd giving an x-coordinate of 1007.
The y-coordinates can be shown to follow the pattern 0,-2,-2,2,2,-4,-4,4,4,-6,-6,....
We notice that for any number of moves "n" that is divisible by 4 that the y-coordinate is $\frac{n}{2}$ and we note that the 2 previous moves have y-coordinate $-\frac{n}{2}$ and the next one has y-coordinate $\frac{n}{2}$.
Then for $n = 2012$ the y-coordinate is $\frac{2012}{2} = 1006$.
(Note. For $n = 2010$ or $n = 2011$ the y-coordinate would be -1006).
For $n = 2013$ the y-cordinate is 1006. After 2013 moves the compulsive ant is at (1007,1006) and is $\sqrt{1007^2 + 1006^2} \doteq 1423.4$.