Intermediate Math Circles
Counting Problem Set
November 6, 2013

Solutions

1. A restaurant menu lists 5 meat dishes and 3 fish dishes.

   a) How many single course dinners can you order?

      Using the Sum Rule you have $5 + 3 = 8$ choices.

   b) How many dinners can you order that have 1 meat dish and 1 fish dish?

      Using the Product Rule you have $5 \times 3 = 15$ choices.

2. How many numbers between 1000 and 9999 have only even digits?

   The numbers all have 4 digits. There are 4 choices for the first digit (2, 4, 6, and 8), and 5 choices for every other digit (0, 2, 4, 6, and 8). Using the Product Rule there are $4 \times 5 \times 5 \times 5 = 500$ numbers.

3. A licence plate consists of 4 letters followed by 3 digits. How many different license plates are possible?

   There are 26 choices for each of the 4 letters, and 10 choices for each of the 3 digits. Using the Product Rule there are $26^4 \times 10^3 = 456\ 976\ 000$ possible licence plates.

4. How many 3 digit numbers are there in which adjacent digits are not the same?

   There are 9 choices for the first digit (0 cannot be used if it is a three digit number). Then for each of these possibilities there are only 9 choices for the second digit because this digit must not be the same as the first. Then for each of these possibilities there are only 9 choices for the third digit since the third digit must not be the same as the second digit. Then, by the product rule, there are $9 \times 9 \times 9 = 729$ three digit numbers in which adjacent digits are not the same.

   Sometimes when solving counting problems the "easiest" method does not come to the solver. What follows is another different solution which is mathematically very correct. If there is a message here, trying something is always better than not trying anything at all.

4. How many 3 digit numbers are there in which adjacent digits are not the same?

   It is possible to use an indirect method to find out how many 3 digit numbers have adjacent digits which are the same and then subtract from the total number of three digit numbers to determine the number of numbers in which no adjacent digits are the same.

   In total, using the product rule, there are $9 \times 10 \times 10 = 900$ three digit numbers.

   In counting the number of numbers in which adjacent digits are the same, there are 3 cases to consider:

   1. All three digits are the same
   2. The first two digits are the same
   3. The last two digits are the same

   The first case can happen in 9 possible ways (111, 222, ..., 999).

   For the second case, there are 9 choices for the first digit (it can't be 0), then 1 choice for the second digit and 10 choices for the last digit. Notice that by not restricting the last digit we have included numbers with three repeated digits.

   For the third case, there are 10 choices for the second digit, 1 choice for the third digit and 9 choices for the first digit. Notice that we have again included the first case.

   Using the Sum and Product Rules, there are $9 \times 1 \times 10 + 9 \times 10 \times 1 = 180$ numbers with adjacent digits the same but we have counted case 1 twice. So we subtract 9 from 180 to get 171 numbers with at least two adjacent digits the same. This means that there are $900 - 171 = 729$ three digit numbers in which no adjacent digits are not the same.

5. In how many was can 6 people seat themselves in a room with 9 chairs where at most 1 person can sit in each chair?

   The first person has 9 choices as to where they may sit.
   The second person has 8 choices.
   The third person has 7 choices.
   The fourth person has 6 choices.
   The fifth person has 5 choices.
   The sixth person has 4 choices.

   Using the Product Rule, there are $9 \times 8 \times 7 \times 6 \times 5 \times 4 = 60\ 480$ ways for the people to seat themselves.

6. How many permutations of the numbers 1, 2, 3, 4, 5, and 6:

   a) begin with an even number?

      There are 3 choices for the first number, and then it is simply a matter of permuting the remaining five numbers.

      Using the Product Rule there is a total of $3 \times 5! = 360$ permutations.

   b) begin with an odd number and end with an even number?

      There are 3 choices for the first number, 3 choices for the last number, and then we permute the remaining 4 numbers in the middle spaces.

      Using the Product Rule there are $3 \times 4! \times 3 = 216$ total permutations.

   c) begin with an odd number and end with an odd number?

      There are 3 choices for the first number and 2 choices for the last number. The last 4 numbers are then again permuted in the middle spaces.

      Using the Product Rule there is a total of $3 \times 4! \times 2 = 144$ permutations.

7. How many permutations of the numbers 1, 2, 3, 4, 5, 6, 7, and 8 taken 5 at a time:

   a) have 7 and 8 in adjacent positions?

      Begin by grouping 7 and 8 together as one item, giving us 7 items to arrange in a group of 5. To make sure that $\{7, 8\}$ is included in the permutation, put it anywhere in the permutation (there are 4 places it can go), and then permute the 6 remaining numbers into the 3 empty spaces, which can be done in $6 \times 5 \times 4 = 120$ ways.

      Using the Product Rule gives us $4 \times 120 = 480$ permutations, but remember that the grouped item $\{7, 8\}$ can itself be arranged in 2 ways, so the total number of permutations is $2 \times 480 = 960$.

   b) have 7 and 8 separated by exactly 1 number?

      Start by finding how many permutations with 7 appearing before 8 satisfy this condition, then multiply your answer by 2 like in part b) to include permutations with 8 appearing first.
      7 is able to go in the first, second, or third position with 8 appearing in the third, fourth, or fifth position respectively. The 6 remaining digits can then be permuted in the 3 remaining spaces.

      In total - using the Product Rule and brackets for clarity - there are
      $3 \times (6 \times 5 \times 4) \times 2 = 720$ possible permutations.