Intermediate Math Circles
November 6, 2013
Counting I

Counting in mathematics is determining the number of ways in which something can occur. For example, counting how many license plates are possible using letters and numbers in a certain way, finding the number of possible ways of getting a certain hand in a game of cards, or counting how many possible combinations there are on a lock.

The study of the method for counting these kinds of things is called **combinatorics**.

The first thing we will learn, does not simply apply to combinatorics, but to all problem solving. The fundamental trick of problem solving is to take a hard problem and turn it into several easier problems.

In combinatorics, two important rules which will help us to count are the **product rule** and **sum rule**.

Product Rule

If a first action can be performed in p ways and a second action can be performed in q ways, then the two actions can be performed together in $p \times q$ ways. The Product Rule for counting is also referred to as ***The Fundamental Counting Principle***.

Example 1:

Cal Q. Lator has 5 different shirts and 3 different pairs of pants. How many different outfits can he wear?

Solution:

Since there are 5 ways for Cal to select a shirt and 3 ways for Cal to select a pair of pants, the product rule tells us that there are 5×3 or 15 different possible outfits.

Of course, we could have also solved this problem by making a table which lists all of the possibilities. Let us do that to check our answer.

Let s_1, s_2, s_3, s_4 and s_5 represent the shirts.
Let p_1, p_2, and p_3 represent the pants.

$s_1\ p_1$	$s_2\ p_1$	$s_3\ p_1$	$s_4\ p_1$	$s_5\ p_1$
$s_1\ p_2$	$s_2\ p_2$	$s_3\ p_2$	$s_4\ p_2$	$s_5\ p_2$
$s_1\ p_3$	$s_2\ p_3$	$s_3\ p_3$	$s_4\ p_3$	$s_5\ p_3$

$\therefore$ there are 15 different possible outfits for Cal to wear.

Example 2:

A company makes both 1" and 2" binders which come in four colours: red, green, black or blue. How many different binders do they make?

Solution:

They have 2 sizes and 4 different colours, so the product rule tells us that there are $2 \times 4 = 8$ different possible makes of binder.

Sum Rule:

If a certain action can be performed in p ways and another action can be performed in q ways and neither action can be performed at the same time, then there are $p + q$ ways in which either the first action or second action can be performed.

Example 3:

A company makes both 1" and 2" binders which come in four colours: red, green, black or blue. How many different binders do they make? Solve this problem using the sum rule.

Solution:

We are looking at example 2 in a different way. In the first group the company makes 1" binders which are red, green, black or blue. In the second group the company makes 2" binders which are red, green, black, or blue. Hence, the first type (1" binders) has 4 objects as does the second type (2" binders). Thus the sum rule gives us that there are $4 + 4 = 8$ total objects. A binder cannot be both 1" and 2" at the same time.

Example 4:

How many two digit positive numbers end in a 5 or 7?

Solution:

What are the two cases?

Case 1: The number ends in a 5. The numbers have the form $\square 5$. There are 9 choices for the first digit (1,...,9), and one choice for the second digit, namely 5. Hence there are $9 \times 1 = 9$ two digit numbers that end in 5. Note that the first digit cannot be zero if the number is a two digit number.

Case 2: The number ends in a 7. The numbers have the form $\square 7$. There are 9 choices for the first digit (1,...,9), and one choice for the second digit, namely 7. Hence there are $9 \times 1 = 9$ two digit numbers that end in 7. Note that the first digit cannot be zero if the number is a two digit number.

Thus, by the sum rule, there are $9 + 9$ or 18 two digit numbers which end in a 5 or 7.

Note: The product rule and the sum rule may be extended to the case where there are more than 2 actions or groups.

Example 5:

A restaurant has 4 appetizers, 3 main courses, and 5 desserts. How may different meals consisting of one appetizer, one main course and one dessert could you order?

Solution:

There are 4 ways of selecting an appetizer, 3 ways of selecting a main course and 5 ways of selecting a dessert, so using the product rule, there are $4 \times 3 \times 5 = 60$ different possible meals.

Sometimes in counting problems it is easy to "twist" problems by changing the wording ever-so-slightly.

Example 6:

A restaurant has 4 appetizers, 3 main courses, and 5 desserts. How may different meals consisting of at least one item could you order? You are not allowed to have more than one item from a particular group.

Solution:

Now there are 5 ways of selecting an appetizer (the four different ones plus the choice of no appetizer at all), 4 ways of selecting a main course (the three different ones plus the choice of no main course at all), and 6 ways of selecting a dessert (the five different ones plus the choice of no dessert at all). So, using the product rule, there are $5 \times 4 \times 6 = 120$ different possible meals. But this includes the possibility of no meal at all. Therefore, we subtract 1 from 120 and get 119 meals consisting of at least one item.

Example 7:

How many positive numbers less than 1000 have 1 as the first digit?

Solution:

There are 3 cases: three digit numbers which have a first digit one, two digit numbers which have a first digit one and one digit numbers which have a first digit one. We will use the product rule for each case to determine the number of numbers in that particular case. Then, since there is no overlap between cases, we will use the sum rule to determine the overall number of possible numbers.

Case 1: Three digit numbers of the form $1\square\Diamond$. The first digit must be 1, there are 10 choices for $\square$ and 10 choices for $\Diamond$. Hence, by the product rule, there are $1 \times 10 \times 10$ or 100 possible three digit numbers whose first digit is one.

Case 2: Two digit numbers of the form $1\square$. There are 10 choices for $\square$ and the first digit must be a one. Hence, by the product rule, there are 1×10 or 10 possible two digit numbers whose first digit is one.

Case 3: There is only 1 one digit number whose first digit is 1.

Using the sum rule, there are $100 + 10 + 1$ or 111 numbers less than 1000 which have 1 as the first digit.

Example 8:

How many 2 digit positive numbers are divisible by either 2 or 5?

Solution:

We have two groups of numbers: those divisible by 2 and those divisible by 5. We cannot apply the sum rule because the sum rule says that we must have no objects which are in both groups. In this problem we have some numbers that are divisible by both 2 and 5. Thus, we have to make sure that we do not count these numbers twice.

Case 1: Two digit numbers divisible by 2. These numbers have the form $\Box\Diamond$ where $\Box$ has 9 possibilities (the first digit cannot be zero) and $\Diamond$ has 5 possibilities (0,2,4,6,8) because only even numbers are divisible by 2. Thus, by the product rule, this case has 9×5 or 45 numbers.

Case 2: Two digit numbers divisible by 5 but not divisible by 2. These numbers have the form $\Box\Diamond$ where $\Box$ has 9 possibilities (the first digit cannot be zero) and $\Diamond$ has only 1 possibility (5) because only numbers ending in 0 or 5 are divisible by 5 but we have already counted the numbers ending in 0 in the first case. Thus, by the product rule, this case has 9×1 or 9 numbers.

Using the sum rule, there are $45 + 9$ or 54 numbers that are divisible by 2 or 5.

Note: Another way of doing this problem would be add the number of numbers divisible by 2 and the number of numbers divisible by 5 and then subtract the number of numbers divisible by both 2 and 5 which have been counted twice, once in each group.

Solution:

We have three groups of numbers: those divisible by 2, those divisible by 5, and those divisible by both 2 and 5.

Case 1: Two digit numbers divisible by 2. These numbers have the form $\Box\Diamond$ where $\Box$ has 9 possibilities (the first digit cannot be zero) and $\Diamond$ has 5 possibilities (0,2,4,6,8) because only even numbers are divisible by 2. Thus, by the product rule, this case has 9×5 or 45 numbers.

Case 2: Two digit numbers divisible by 5. These numbers have the form $\Box\Diamond$ where $\Box$ has 9 possibilities (the first digit cannot be zero) and $\Diamond$ has only 2 possibilities (0,5) because only numbers ending in 0 or 5 are divisible by 5. Thus, by the product rule, this case has 9×2 or 18 numbers.

Case 3: Two digit numbers divisible by 5 and 2. These numbers have the form $\Box\Diamond$ where $\Box$ has 9 possibilities (the first digit cannot be zero) and $\Diamond$ has only 1 possibility (0) because, to be divisible by 2 and 5, the number must be divisible by 10 and therefore must end in zero. Thus, by the product rule, this case has 9×1 or 9 numbers.

To determine the number of two digit numbers we add the number of numbers divisible by 5 and the number of numbers divisible by 2. We then subtract the number of numbers divisible by both 5 and 2 which have been counted in both groups. Therefore, by the sum rule, there are $45 + 18 - 9$ or 54 two digit numbers divisible by 2 or 5, as in the first solution.

Excercises

1. Al G. Braw has a 5 math books, 3 science books, and 2 history books.

   a.) If Al wants to read one book, how many choices does he have?

   b.) If Al wants to read one math book, one science book and one history book, how many choices does he have?

2. A vehicle licence plate number consists of 3 letters followed by 3 digits. How many different licence plates are possible?

3. A car can be ordered with 5 choices of colour, 3 choices of upholstery, with or with out air conditioning, with or without cruise control, and with or without sunroof. How many different cars can be ordered?

4. Determine how many 3 digit positive integers are either divisible by 2 or have 3 as the first digit.

5. Create a counting problem with 810 as the answer.

Answers

1a	5+3+2=10
1b	$5 \times 3 \times 2 = 30$
2	$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 26^3 \times 10^3$
3	$5 \times 3 \times 2 \times 2 \times 2 = 120$
4	$9 \times 10 \times 5 + 1 \times 10 \times 5 = 500$
5	Determine the number of three digit numbers which do not have the second digit 5.

Permutations

In many of the examples so far we have been picking different choices from different sets (colour, size, etc). However, in many cases, we want to pick the items from one set in a particular order.

A ***permutation*** of n distinct objects is an arrangement of all of the objects in a $definite\ order$.

Notes:

- Order does matter! In the permutation (or arrangement) of the three digits 1, 2, and 3, 321 is not the same as 132. In cases where we don't care about the order we would instead call them ***combinations***.
- When we say permutations, we mean that each item from the set may only be used once. However, in some cases we may want to use the items more than once and so we call these ***permutations with repetition***.

List some examples where we would want to use permutations? combinations? permutations with repetition?

Permutations: In how many ways can three potted plants be arranged on a window sill?

Combinations: Picking lottery numbers.

Permutations with repetition: selecting a combination on a combination lock (should be called a permutation lock, since order matters!)

For now, we will just look at permutations and we will look at combinations in the future.

Example 9:

How many permutations are there on n objects?

Solution:

Out of the n objects, we need to pick one object to be first. We have n choices for this. Then, since we are not using repetitions, we now only have $(n-1)$ objects left to choose from so we have $(n-1)$ choices for the second object. Then, since we are not using repetitions, we now only have $(n-2)$ objects left to choose from so we have $(n-2)$ choices for the third object. Continuing, we see that by the product rule we get that there are $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ permutations on n objects.

Since this product occurs frequently, we give it a special notation called ***factorial notation***.

We write $n!$ (which reads "$n\ factorial$") and $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$.

For example, $3! = 3 \times 2 \times 1 = 6$ and $7! = 7 \times 6 \times 5 \times \cdots \times 2 \times 1 = 5040$.

Example 10:

There are 5 runners in a race. Different prizes are awarded to the runners who finish first, second or third. In how many ways can the prizes be distributed?

Solution:

In this problem we see that we are not looking to count all possible permutations, but rather just the permutations where we select 3 of the 5 runners.

Let the possible distribution of prizes be represented by abc, where a is the runner who finishes first, b is the runner who finishes second and c is the runner who finishes third. Observe that there are 5 possible runners who could finish first, so there are 5 choices for a. After one runner finishes, there are only 4 remaining who could finish second so we have 4 choices for b. Similarly, we have 3 choices for c. So the number of possible ways of distributing the prizes is $5 \times 4 \times 3$ or 60.

When we are not counting all permutations, but just counting the number of permutations in which we select k objects from n objects, we call these permutations on n objects taken k at a time.

Excercises

1. Consider the set $\{a, b, c, d\}$.

   a.) How many permutations are there on the set? List them.

   b.) How many permutations on the set have a as the first letter? List them.

   c.) How many permutations on the set have b and c together? List them.

2. How many permutations are there on the numbers 1 to 5 taken two at a time. List them.

3. A student club with 10 members wishes to select a president, a secretary and a treasurer from its membership. No member may be selected for more than one office. In how many ways can this be done?

4. In how many ways can 25 students be seated in a classroom (a) with 25 desks? (b) with 30 desks?

Answers

1a	$4! = 24$
1b	$3! = 6$
1c	$2 \times 3!$
2	$5 \times 4 = 20$
3	$10 \times 9 \times 8 = 720$
4	(a) $25!$ (b) $30 \times 29 \times 28 \times \cdots \times 4$

Problem Set

1. A restaurant menu lists 5 meat dishes and 3 fish dishes.

   a.) How many single course dinners can you order?

   b.) How many dinners can you order than have 1 meat dish and 1 fish dish?

2. How many numbers between 1000 and 9999 have only even digits?

3. A licence plate consists of 4 letters followed by 3 digits. How many different license plates are possible?

4. How many 3 digit numbers are there in which adjacent digits are not the same?

5. In how many ways can 6 people seat themselves in a room with 9 chairs where at most 1 person can sit in each chair?

6. How many permutations of the numbers 1, 2, 3, 4, 5, and 6:

   a.) begin with an even number?

   b.) begin with an odd number and end with an even number?

   c.) begin with an odd number and end with an odd number?

7. How many permutations of the numbers 1, 2, 3, 4, 5, 6, 7, and 8 taken 5 at a time:

   a.) have 7 and 8 in adjacent positions

   b.) have 7 and 8 separated by exactly 1 number