Intermediate Math Circles
Problem Set Solutions
From October 9, 2013

1. The length and width of the Canadian flag are in the ratio 2 : 1. A company buys a flag with area 10 m^2. What are the dimensions of the flag?

   Let x represent the width of the flag and $2x$ represent the length of the flag.

$$\begin{aligned} \text{Area} &= \textit{Length} \times \textit{Width} \\ 10 &= (2x)(x) \\ 10 &= 2x^2 \\ 5 &= x^2 \\ \therefore x &= \sqrt{5},\ x > 0 \end{aligned}$$

   Since $x = \sqrt{5}$, $2x = 2\sqrt{5}$ and the dimensions of the flag are $2\sqrt{5}$ m by $\sqrt{5}$ m (approximately 4.47 m by 2.24m).

2. A right triangle with sides in the ratio 3 : 4 : 5 has one side of length 60 mm. What are the possible lengths of the other sides?

   The side of length 60 mm could be the smallest side, the largest side or the third side (neither the largest nor the smallest).

   Let the smallest side be 60 mm and the other two sides be y_1 and z_1.
   Then $3 : 4 : 5 = 60 : y_1 : z_1$. Since we can multiply 3 by 20 to get 60, we multiply the other two terms of the ratio by 20 to obtain $y_1 = 80$ mm and $z_1 = 100$ mm.

   Let the middle side be 60 mm and the other two sides be x_2 and z_2.
   Then $3 : 4 : 5 = x_2 : 60 : z_2$. Since we can multiply 4 by 15 to get 60, we multiply the other two terms of the ratio by 15 to obtain $x_2 = 45$ mm and $z_2 = 75$ mm.

   Let the largest side be 60 mm and the other two sides be x_3 and y_3.
   Then $3 : 4 : 5 = x_3 : y_3 : 60$. Since we can multiply 5 by 12 to get 60, we multiply the other two terms of the ratio by 12 to obtain $x_3 = 36$ mm and $y_3 = 48$ mm.

   $\therefore$ there are three possible right triangles with sides in ratio 3:4:5 with one side 60 mm. The triangles are {60mm, 80mm, 100 mm}, and {45 mm, 60 mm, 75 mm}, and {36 mm, 48 mm, 60 mm}.

3. Emily's chemistry teacher instructed her class to make a saltwater solution with a 2 : 3 ratio of salt to water. Emily misheard her teacher and made a 100 mL solution with a 2 : 3 ratio of water to salt instead. How much water must she add to have the correct solution?

   Let the present amount of water be $2x$ mL and the present amount of salt be $3x$ mL. The total amount of mixture is $2x + 3x = 5x$ mL.
   $\therefore$ $5x = 100$ and $x = 20$. So at present there is $2x = 2(20) = 40$ mL of water and $3x = 3(20) = 60$ mL of salt.

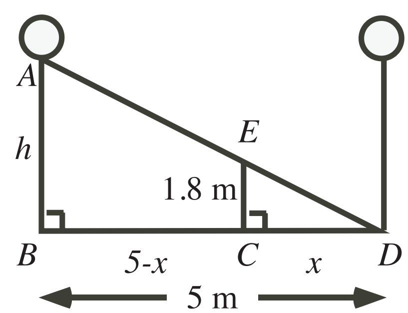
   Since the required ratio of water to salt is 3:2, the ratio of water to the entire mixture is 3:5. Let w represent the amount of water added to the 100 mL of mixture. There is now $(40 + w)$ mL of water and $(100 + w)$ mL of mixture.

$$\begin{aligned} \therefore \frac{40+w}{100+w} &= \frac{3}{5} \\ 5w + 200 &= 3w + 300 \\ 2w &= 100 \\ w &= 50 \end{aligned}$$

   $\therefore$ by adding 50 mL of water to the present mixture Emily creates the desired ratio.

4. Evan is 1.8 metres tall. He walks between two lampposts that are 5 metres apart and notices that his shadow from one of the lampposts just touches the base of the other. He also notices that the ratio of his height to the height of the lamppost is 2 to 5. How far is he from each lamppost?

   Draw a diagram to represent the problem. Let h represent the height of the lamppost and x represent the length of the shadow.



   Since we know the ratio of Evan's height to the height of the lamppost, $\frac{1.8}{h} = \frac{2}{5}$. Then $2h = 9$ and $h = 4.5$ m. The lamppost is 4.5 m tall.

   Now $\triangle ADB \sim \triangle EDC$ since $\angle D$ is common and $\angle ABD = \angle ECD = 90°$. $\therefore$ $\frac{AB}{EC} = \frac{BD}{CD}$ and $\frac{4.5}{1.8} = \frac{5}{x}$.

   It follows that $4.5x = 9$ and $x = 2$ m.

   $\therefore$ Evan is 2 m from one lamppost and $5 - 2 = 3$ m from the other lamppost.

5. A wildlife observer captures 40 deer from a forest, tags them and lets them go back in the forest. Later, he captures 60 deer from the same forest and sees that 18 of them are tagged. What is the approximate number of deer living in the forest? We assume that the ratio of tagged deer to untagged deer in the captured group is the same as the ratio in the entire forest.

   Let u represent the approximate number of deer in the forest.

$$\begin{aligned} \text{Tagged : Total} = 40 : u &= 18 : 60 \\ \frac{40}{u} &= \frac{18}{60} \\ 18u &= 2400 \\ u &= \frac{400}{3} \doteq 133 \end{aligned}$$

   There are approximately 133 deer in the forest.

6. In a room, there are a number of people and a number of dogs. The ratio of heads to legs in the room is 5 : 13. Assuming that there are no amputees present, what is the ratio of humans to dogs in the room?

   Let h represent the number of humans.
   Let d represent the number of dogs.

   The number of heads in the room is $(h + d)$.
   The number of legs in the room is $(2h + 4d)$.

   We know that the ratio of heads to legs is 5:13 so

$$\begin{aligned} \frac{h+d}{2h+4d} &= \frac{5}{13} \\ 13h + 13d &= 10h + 20d \\ 3h &= 7d \\ \frac{h}{d} &= \frac{7}{3} \end{aligned}$$

   $\therefore$ the ratio of the number of humans to the number of dogs in the room is 7:3.

7. Two schools are getting together for a dance. The first school has a $3:5$ ratio of boys to girls. The second school has a $4:3$ ratio of boys to girls. All of the students attend the dance and during one song each boy is paired up with one girl. What is the ratio of the number of students at the first school to the number of students at the second school?

   Let the number of boys at the first school be $3p$ and the number of girls be $5p$. Therefore there are $8p$ students at the first school.

   Let the number of boys at the second school be $4q$ and the number of girls be $3q$. Therefore there are $7q$ students at the second school.

   During one dance every boy was paired up with one girl. Therefore the total number of boys equals the total number of girls. $\therefore 3p + 4q = 5p + 3q$ and $q = 2p$.

   Substituting $2p$ for q, the second school has $7q = 7(2p) = 14p$ students.

   Since the first school has $8p$ students and the second school has $14p$ students, the required ratio is $8p : 14p = 4 : 7$.

   $\therefore$ the ratio of the number of students at the first school to the number of students at the second school is $4 : 7$.

8. Kaelee wants to make 75% cocoa chocolate. To do this she mixes together melted 60% cocoa chocolate and 85% melted cocoa chocolate. If she uses 200 mL of the 60% cocoa chocolate, how much 85% chocolate will she need to use?

   In 200 mL of melted 60% cocoa chocolate there is $0.6 \times 200 = 120$ mL of cocoa chocolate.

   Let x represent the number of mL of 85% cocoa chocolate required. In the 85% cocoa chocolate there is $0.85x$ mL of cocoa chocolate.

   We require a mixture that is 75% cocoa chocolate.

$$\begin{aligned} \frac{\text{Amount of Cocoa Chocolate}}{\text{Total Mixture}} &= 0.75 \\ \frac{120 + 0.85x}{200 + x} &= 0.75 \\ 120 + 0.85x &= 150 + 0.75x \\ 0.10x &= 30 \\ x &= 300 \end{aligned}$$

   $\therefore$ 300 mL of 85% cocoa chocolate must be added to make the 75% cocoa chocolate mixture.

9. Quick Shipping Co. will only ship boxes that have the sum of their length, width and height at most 240 cm. If a box has its length, width and height in the ratio 3 : 2 : 5, and it is small enough to be shipped, what is its maximum possible volume?

   Since the ratio of the length to width to height is 3 : 2 : 5, let the length be $3x$, the width be $2x$, and the height be $5x$. Then $3x + 2x + 5x \leq 240$ or $10x \leq 240$ and $x \leq 24$. We want the largest value of x so that each of the dimensions is the largest. It then follows that $x = 24$.

   Then the length is $3x = 3(24) = 72$ cm, the width is $2x = 2(24) = 48$ cm, and the height is $5x = 5(24) = 120$ cm.

   The volume is calculated by multiplying the length, width and height. The maximum volume is $72 \times 48 \times 120 = 414\ 720$ cm$^3 \doteq 0.41$ m^3.

   $\therefore$ the largest box whose dimensions are in the ratio 3 : 2 : 5 has volume 414 720 cm^3 (approximately $\doteq 0.41$ m^3).

10. Kool-Aid is made with flavouring mix, sugar and water. The directions instruct you to mix the flavouring mix, sugar and water in the ratio 1 : 8 : 32. Jackie decides to make Kool-Aid. She adds 30 mL of flavouring mix and 240 mL of sugar to a 1000 mL bottle, then fills the rest with water. She wants to drink some of this extra sugary Kool-Aid, and then re-fill the bottle with water to bring it to the proper ratio. How much will she need to drink?

    The 1 000 mL container has 30 mL of flavour mix, 240 mL of sugar and $1000 - 30 - 240 = 730$ mL of water. In fractional terms, $\frac{3}{100}$ is flavour mix, $\frac{24}{100}$ is sugar and $\frac{73}{100}$ is water.

    Let the amount Jackie drinks be x mL. (This would also be the amount of water she would have to add.)

    By drinking x mL, she removes $\frac{3}{100}x$ mL of flavouring leaving $(30 - \frac{3}{100}x)$ mL of flavouring, $\frac{24}{100}x$ mL of sugar leaving $(240 - \frac{24}{100}x)$ mL of sugar and $\frac{73}{100}x$ mL of water leaving $(730 - \frac{73}{100}x)$ mL of water. Jackie then adds x mL of water resulting in $(730 - \frac{73}{100}x + x)$ mL of water.

    Since the ratio of flavouring to sugar to water is 1:8:32,

$$\frac{30 - \frac{3}{100}x}{1} = \frac{240 - \frac{24}{100}x}{8} = \frac{730 - \frac{73}{100}x + x}{32}$$

    The second term in the ratio is 8 times the first so we will work with the first and third terms. Multiplying by 32 and converting the remaining fractions to decimals,

$$\begin{aligned} 32(30 - 0.03x) &= (730 - 0.73x + x) \\ 960 - 0.96x &= 730 + 0.27x \\ 230 &= 1.23x \\ x &\doteq 187 \end{aligned}$$

    $\therefore$ Jackie should drink approximately 187 mL of the extra sugary drink. If she then adds 187 mL of water the correct mix ratio will be attained.

11. (AMC 12A 2003) Al, Bert and Carl are to divide a pile of Halloween candy in the ratio $3:2:1$ because they placed first, second and third, respectively, in a draw at their school. Due to some confusion, they come at different times to claim their prize, and each assumes he is the first person to arrive. If each takes what he believes to be the correct share of candy, what fraction of the candy goes unclaimed?

    Al gets 3 parts out of 6 parts of the candy. Therefore Al gets $\frac{1}{2}$ of the candy that is there and after he has taken his candy he leaves $1 - \frac{1}{2} = \frac{1}{2}$ of the candy.

    Bert gets 2 parts out of 6 parts of the candy. Therefore Bert gets $\frac{1}{3}$ of the candy that is there and after he has taken his candy he leaves $1 - \frac{1}{3} = \frac{2}{3}$ of the candy.

    Carl gets 1 part out of 6 parts of the candy. Therefore Carl gets $\frac{1}{6}$ of the candy that is there and after he has taken his candy he leaves $1 - \frac{1}{6} = \frac{5}{6}$ of the candy.

    Since we are interested in the amount that is left after all three have taken their candy, we can simply look at what each leaves after they take their candy.

    If Al comes first, he leaves $\frac{1}{2} \times 1$.
    If Bert comes next, he leaves $\frac{2}{3} \times \left(\frac{1}{2} \times 1\right)$.
    And if Carl comes last, he leaves $\frac{5}{6} \times \left(\frac{2}{3} \times \left(\frac{1}{2} \times 1\right)\right) = \frac{5}{18}$ of the candy.

    Since multiplication is commutative, the product of what is left will be an arrangement of the product above. It does not matter in what order they arrive and what they leave after all three have taken candy will be the same.

    $\therefore$ after all three have taken candy $\frac{5}{18}$ of the original amount of candy will remain.