Intermediate Math Circles
October 30, 2013
Rates II

Warm-Up

Just a small change in how people vote can affect an election result. In a recent election, the ratio of the number of voters for the Purple Party to the number of voters for the Pink Party was 15:16 and the Pink Party won the election. Had 300 more people voted for the Purple Party and 200 fewer people voted for the Pink Party, the ratio would have been 11:10 and the Purple Party would have won the election. Determine the total number of votes originally cast.

Solution:

Let a represent the number of votes originally cast for the Purple Party.
Let b represent the number of votes originally cast for the Pink Party.
Then the total number of votes originally cast was $a + b$.

The original ratio of votes cast was $a : b = 15 : 16$. This ratio can be written $\frac{a}{b} = \frac{15}{16}$ and $a = \frac{15}{16}b$ follows. (1)

Had 300 more people voted for the Purple Party, the Purple Party would have received $(a+300)$ votes. Had 200 fewer people voted for the Pink Party, the Pink Party would have received $(b-200)$ votes. Then

$$
\begin{aligned}
\frac{a+300}{b-200} &= \frac{11}{10} \\
10a + 3000 &= 11b - 2200 \\
10a &= 11b - 5200 \\
10\left(\frac{15b}{16}\right) &= 11b - 5200 && \text{Substituting for } a \text{ from (1)} \\
5\left(\frac{15b}{8}\right) &= 11b - 5200 \\
75b &= 88b - 41600 && \text{Multiplying by 8} \\
-13b &= -41600 \\
b &= 3\,200 \\
a &= \frac{15}{16}(3200) && \text{Substituting } b = 3200 \text{ in (1)} \\
a &= 3\,000 \\
a + b &= 3\,000 + 3\,200 \\
a + b &= 6\,200
\end{aligned}
$$

There were 6 200 total votes originally cast.

Problem #7 from last week:

Two people are running laps around a 400 m track. They begin at the same point and run in the same direction. The faster person runs at a pace of 1 kilometer every 4 minutes and the slower person runs at a pace of 1 kilometer every 6 minutes. How long will it take until the faster person laps the slower person?

Solution:

Let t represent the time, in minutes, when the faster person laps the slower person. Let d_1 represent the distance, in metres, run by the faster person and d_2 represent the distance, in metres, run by the slower person.

At time t we want $d_1 = d_2 + 400$. (1)

Using $distance = speed \times time$, $d_1 = \left(\frac{1000 \text{ m}}{4 \text{ min}}\right)(t \text{ min})$ and $d_2 = \left(\frac{1000 \text{ m}}{6 \text{ min}}\right)(t \text{ min})$.

Simplifying, $d_1 = 250t$ and $d_2 = \frac{500}{3}t$. We can now substitute into (1) to solve for t.

$$\begin{aligned} 250t = &= \frac{500}{3}t + 400 \\ 750t &= 500t + 1200 \\ 250t &= 1200 \\ t &= \frac{1200}{250} = \frac{24}{5} = 4.8 \end{aligned}$$

$\therefore$ in 4.8 minutes or 4 minutes 48 seconds the faster runner will lap the slower runner.

Problem #9 from last week:

Two horsemen spot each other from 400 m apart, and start riding towards each other, one at 2 $\frac{\text{m}}{\text{s}}$ and the other at 3 $\frac{\text{m}}{\text{s}}$. A fly starts at one horse and, flying at 8 $\frac{\text{m}}{\text{s}}$, flies to the other horse, turns around and immediately flies back. If the fly continues flying back and forth until the horses meet, what total distance does the fly cover? Solution:

Let t represent the time, in seconds, travelled by the two horsemen until they meet.
Since the first horseman is travelling at 2 m/s, he rides $2t$ m. Since the second horseman is travelling at 3 m/s, he rides $3t$ m. Their total distance is 400 m so $2t + 3t = 400$, $5t = 400$ and $t = 80$ seconds.
Since the fly is travelling at 8 m/s, in t seconds it travels $8t$ m. But the fly travels constantly back and forth from one rider to the other until they meet. Therefore the fly is on the move for 80 s. The total distance is $8t = 8(80) = 640$ m.

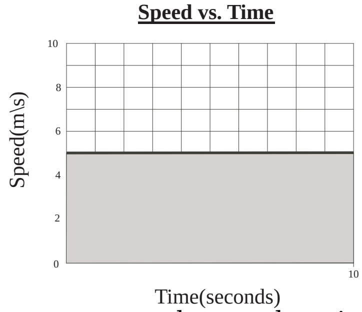
$\therefore$ the fly covers 640 m going back and forth from rider to rider.

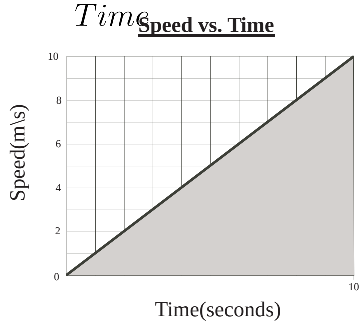
Recap From Last Week:

Acceleration:

Acceleration is the rate at which a moving object slows down or speeds up. This quantity measures a change in speed against time.

$$Acceleration = \frac{Change\ in\ Speed}{Time}$$





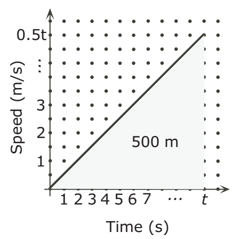
The slope of the line represents the acceleration.

The area under the line corresponds to the distance travelled.

Example

A horse, standing at the start line of a race, begins to run around a 500 m track, accelerating at a rate of 0.5 $\frac{\text{m}}{\text{s}}$. How long does it take the horse to complete the lap?

Solution:



Let t be the time, in seconds, required to complete one 500 m lap of the track.

Then at t seconds, the speed will be $\frac{1}{2}t\ \frac{\text{m}}{\text{s}}$.

The distance is the shaded area under the line.

$$\begin{aligned}
d &= \frac{1}{2} \times \text{Time} \times \text{Speed} \\
500 \text{ m} &= \frac{1}{2} \times (t \text{ s}) \left(\frac{1}{2}t\ \frac{\text{m}}{\text{s}}\right) \\
500 &= \frac{t^2}{4} \\
2000 &= t^2 \\
\therefore\ t &= 20\sqrt{5} \text{ s}, \qquad t > 0
\end{aligned}$$

It takes the horse $20\sqrt{5}$ seconds, approximately 45 seconds, to complete one lap.