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Grade 7 & 8 Math Circles

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Finance

Introduction

A key point in finance is the *time value of money*, a concept which states that "a dollar today is worth more than the same dollar tomorrow". This is because you can *invest* your money and earn *interest*.

Interest

Interest can be seen as a reward for lending your money to a bank or other business. It can also be seen as a price for borrowing money.

Interest rates are expressed as percentages and must be converted into decimals in order to be used in equations. To convert a percentage into a decimal, divide the percentage by 100. This can be simply done using a calculator, or by moving the decimal two places to the left.

$$75\% \longrightarrow 75.0 \longrightarrow 0.75$$

Examples

Convert the following percentages into decimals.

1. 53%	$53 \div 100 = 0.53$	$53\% \longrightarrow 53.0 \longrightarrow 0.53$
2. 8%	$8 \div 100 = 0.08$	$8\% \longrightarrow 08.0 \longrightarrow 0.08$
3. 27.6%	$27.6 \div 100 = 0.276$	$27.6\% \longrightarrow 27.6 \longrightarrow 0.276$
4. 0.6%	$0.6 \div 100 = 0.006$	$0.6\% \longrightarrow 00.6 \longrightarrow 0.006$

To calculate a percentage of a number, multiply the number by the decimal form of the percentage.

Example

Calculate 25% of 700.

$$0.25 \times 700 = 175$$

So 25% of 700 is 175.

Simple Interest

Under simple interest, the interest earned (or paid) is calculated on the original amount invested (or borrowed), called the *principal*, during the whole time of the loan at the stated annual interest rate.

The formula to calculate simple interest, I, is

$$\boxed{I = Prt}$$

where P is the **p**rincipal of the loan, r is the interest **r**ate, and t is the **t**ime in years.

Example

What is the simple interest due on a \$2500 loan at the end of 10 months if the annual interest rate is 7.5%?

$I = ?$

$P = 2500$

$r = 0.075$

$t = \frac{10}{12}$

$$\begin{aligned} I &= Prt \\ &= (2500)(0.075)\left(\frac{10}{12}\right) \\ &= 156.25 \end{aligned}$$

So the total interest due at the end of 5 years is \$156.25.

The *accumulated value*, denoted by S, is the total amount of a loan or investment after t years. Under simple interest, the accumulated value is the sum of the principle amount and the interest amount.

$$S = P + I$$

Since $I = Prt$, this can be written as

$$S = P + Prt$$
$$\boxed{S = P(1 + rt)}$$

Example

I deposit \$5000 into a savings account which earns 6% simple interest annually. How much money will be in the account in ten years?

$S = ?$

$P = 5000$

$r = 0.06$

$t = 10$

$$\begin{aligned} S &= P(1+rt) \\ &= 5000[1+(0.06)(10)] \\ &= 5000[1+0.6] \\ &= 8000 \end{aligned}$$

The savings account has \$8000 in it after ten years.

What if you are asked to solve a problem where you are given the accumulated value, but not the principal amount? This is called a *present value calculation* and all you must do is rearrange the equation to solve for P instead of S. As a matter of fact, you can use your equation solving skills to solve for any unknown.

Example

How much would I have to deposit into my savings account today in order to have \$1600 in 3 years if the account earns 5% simple interest annually?

$S = 1600$

$P = ?$

$r = 0.05$

$t = 3$

$$\begin{aligned} S &= P(1+rt) \\ \frac{S}{(1+rt)} &= \frac{P(1+rt)}{(1+rt)} \\ P &= \frac{S}{(1+rt)} \\ &= \frac{1600}{(1+(0.05)(3))} \\ &= \frac{1600}{1.15} \\ &= 1391.30 \end{aligned}$$

I would have to deposit \$1391.30 in the savings account today.

Compound Interest

Under compound interest, the interest earned (or paid) is calculated on the original amount invested (or borrowed) and the interest already earned (or paid), during the whole time of the loan at the stated annual interest rate.

Example

Assume that you have deposited \$1000 in a savings account that pays 10% interest compounded annually.

After one year, the money earns $1000 \times 0.1 = \$100$. The new balance is \$1100.
After two years, the money earns $1100 \times 0.1 = \$110$. The new balance is \$1210.
After three years, the money earns $1210 \times 0.1 = \$121$. The new balance is \$1331.
And so on.

Under compound interest, the formula to calculate accumulated value, S, is

$$\boxed{S = P(1+r)^t}$$

where P is the **p**rincipal of the loan, r is the interest **r**ate, and t is the **t**ime in years.

Examples

1. I deposit \$5000 into a savings account which earns 6% interest compounded annually. How much money will be in the account in ten years?

   $S = ?$
   $P = 5000$
   $r = 0.06$
   $t = 10$

   $$\begin{aligned} S &= P(1+r)^t \\ &= 5000(1+0.06)^{10} \\ &= 8954.24 \end{aligned}$$

   The savings account has \$8954.24 in it after ten years.

2. How much do I have to deposit today in order to have \$5000 in 5 years if interest is 8.5% compounded annually?

   $S = 5000$
   $P = ?$
   $r = 0.085$
   $t = 5$

   $$\begin{aligned} \frac{S}{(1+r)^t} &= P \\ \frac{5000}{(1+0.085)^5} &= \\ 3325.23 &= \end{aligned}$$

   I should deposit \$3325.23 today in order to have \$5000 in 5 years.

Nominal Rates

Thus far we have only used compound interest rates that are compounded annually. The term "compounded annually" means that the interest is calculated and applied once a year. More frequent periods are also available. For example, interest could be calculated twice a year, four times a year, every month, every week, every day, etc. Because of this, we must make a couple of small changes to our formula for the accumulated value of a principal amount P:

$$\boxed{S = P\left(1 + \frac{r^{(m)}}{m}\right)^n}$$

where m is the number of compounding periods in a year, $r^{(m)}$ is the nominal rate of interest compounded m times a year, and $n = mt$ is the number of compounding periods over the entire life of the loan.

In the context of a question, how do you know what m is?

Ask yourself: "how many compounding periods can fit in one year?"

"compounded annually"	$\Rightarrow$	$m = 1$
"compounded semi-annually"	$\Rightarrow$	$m = 2$
"compounded quarterly"	$\Rightarrow$	$m = 4$
"compounded monthly"	$\Rightarrow$	$m = 12$
"compounded weekly"	$\Rightarrow$	$m = 52$
"compounded daily"	$\Rightarrow$	$m = 365$

Example

I deposit \$5000 into a savings account which earns 6% interest compounded monthly. How much money will be in the account in ten years?

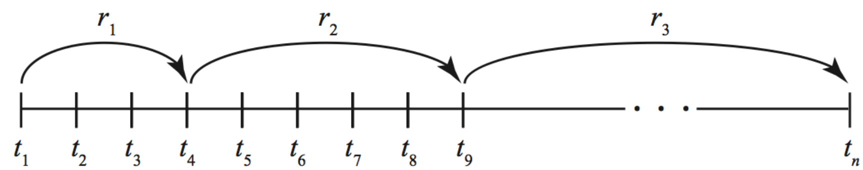
$S =?$
$P = 5000$
$m = 12$
$r^{(m)} = r^{(12)} = 0.06$
$n = 12 \times 10 = 120$

$$\begin{aligned} S &= P\left(1 + \frac{r^{(m)}}{m}\right)^n \\ &= 5000\left(1 + \frac{0.06}{12}\right)^{120} \\ &= 5000(1 + 0.005)^{120} \\ &= 9096.98 \end{aligned}$$

The savings account has \$9096.98 in it after ten years.

Varying Rates

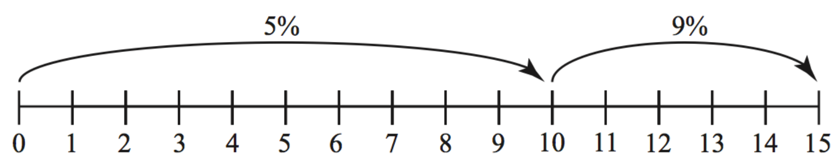
Usually interest rates do not remain the same over the entire life of a loan. To deal with this we don't need to learn any new techniques, but number lines can be very useful to keep all the information organized.



By labelling the compounding periods along the bottom and the changing rates along the top you can easily represent the question visually.

Example

I borrow \$5000 from a financial institution which charges 5% interest compounded annually for the first ten years and 9% compounded annually every year after that. How much money do I have to repay in 15 years?



$$\begin{aligned}\text{Accumulated value after 10 years} &= 5000(1+0.05)^{10} \\ &= 8144.47 \\ \text{Accumulated value after 15 years} &= 8144.47(1+0.09)^{5} \\ &= 12531.28\end{aligned}$$

This is equivalent to

$$\begin{aligned}\text{Accumulated value after 15 years} &= 5000(1+0.05)^{10}(1+0.09)^{5} \\ &= 12531.28\end{aligned}$$

I have to repay \$12531.28 in fifteen years.

Problem Set

1. Determine the accumulated values of the following loans.
   (a) A \$1200 loan for 7 months at 5% simple interest.
   (b) An \$8000 loan for 4 years at 12.5% simple interest.
   (c) A \$500 loan for 99 days at 10% simple interest. (Note: there are 365 days in a year.)
   (d) A \$750 loan for 15 weeks at 13.25% simple interest. (Note: there are 52 weeks in a year.)
2. Determine the principal value if a savings account holds \$3600 after 10 years at 8% simple interest.
3. A loan of \$100 is to be repaid with \$120 at the end of 10 months. What is the annual simple interest rate?
4. How long will it take \$3000 to earn \$60 interest at 6% simple interest?
5. Determine the accumulated values of the following loans.
   (a) A \$2000 loan for 4 years at 5% interest compounded annually.
   (b) A \$100 loan for 25 years at 7.5% interest compounded annually.
6. Determine the principal value:
   (a) If a savings account holds \$7500 after 10 years at 8% interest compounded annually.
   (b) If a savings account holds \$25000 after 50 years at 4.5% interest compounded annually.
7. Determine what amount must be invested at a rate of 5% to accumulate $S = \$5000$ at the end of four years under
   (a) simple interest;
   (b) compound interest (compounded annually).
8. Determine the accumulated values of the following loans.
   (a) A \$1000 loan for 3 years at 13% interest compounded weekly.

(b) A \$500 loan for 25 years at 4% interest compounded semi-annually.

9. Determine the principal value:

   (a) If a savings account holds \$6000 after 10 years at 15% interest compounded quarterly.

   (b) If a savings account holds \$25000 after 50 years at 12% interest compounded monthly.

10. *A student owes \$400 in 3 months and \$500 in 12 months. What single payment now will repay these debts if the interest rate is 12% compounded quarterly for the first 6 months and 10% compounded monthly afterwards?

11. *Andrew borrows \$1000 from a bank that charges 10% interest compounded quarterly. He chooses to repay his debt with a payment of \$200 in three months and three equal payments of $\$X$ in 6, 9, and 12 months. Determine X.

12. Determine the accumulated values of the following loans.

    (a) A \$1000 loan for 10 years if interest is 4% compounded annually for the first three years, 6% compounded annually for the next three years, and 7% compounded annually for every year after that.

    (b) A \$500 loan for 2 years if interest is 6% compounded monthly for the first year and 13% compounded weekly for the second year.

13. Determine the principal value:

    (a) If a savings account holds \$1000 after 25 years if interest is 5% compounded annually for the first ten years, 7% compounded annually for the next ten years, and 8% compounded semi-annually for every year after that.

    (b) If a savings account holds \$1000 after 5 years if interest is 13% compounded weekly for the first three years and 6% compounded monthly for every year after that.

Answers

Problem Set

1. (a) \$1235 (b) \$12000 (c) \$513.56 (d) \$778.67
2. \$2000
3. 24%
4. 4 months
5. (a) \$2431.01 (b) \$609.83
6. (a) \$3473.95 (b) \$2767.74
7. (a) \$4166.67 (b) \$4113.51
8. (a) \$1476.26 (b) \$1345.79
9. (a) \$1376.03 (b) \$63.84
10. \$836.75
11. \$288.86
12. (a) \$1756.11 (b) \$604.44
13. (a) \$210.83 (b) \$600.97