

Grade 7 & 8 Mathematical Circles

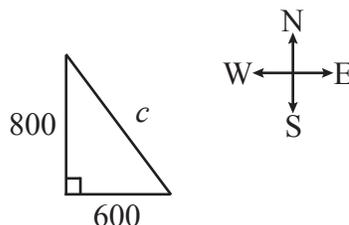
November 12/13, 2013

Mathematical Gems - Solutions

1. If Johnny walks 600 m west and 800 m north, how far away from is he from his starting point?
Give your answer in metres.

Solution:

The first thing we should do is draw a picture!



In the diagram, the variable c represents the total distance Johnny has travelled (in metres). Since the directions west and north are perpendicular, we can form a *right-angled* triangle as seen above.

Because we are dealing with a right-angled triangle, we can use the Pythagorean Theorem!
Thus,

$$\begin{aligned}c^2 &= 600^2 + 800^2 \\ &= 360000 + 640000 \\ &= 1000000\end{aligned}$$

Therefore,

$$c = 1000 \quad \text{or} \quad c = -1000$$

But c is a length, so it cannot be negative. Hence $c = 1000$, and so Johnny is 1000 m from his starting point.

Note: We also could have recognized this triangle as similar to the 6, 8, 10 triangle (a Pythagorean triple) by SAS. Then it must be concluded that

$$\frac{c}{10} = \frac{600}{6} = \frac{800}{8}$$

Thus, it can be shown that $c = 1000$, as we have already done.

2. (a) What is the sum of the integers from 1 to 187?

Solution:

$$n = 187 \implies S = \frac{(187)(188)}{2} = 17578$$

- (b) What is the sum of the integers from 1 to 50?

Solution:

$$n = 50 \implies S = \frac{(50)(51)}{2} = 1275$$

- (c) What is the sum of the integers from 51 to 187?

Solution:

$$\frac{(187)(188)}{2} - \frac{(50)(51)}{2} = 17578 - 1275 = 16303$$

3. Evaluate the sum $\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \dots + \frac{271}{2}$

Solution:

Notice that each term in the sum has a common denominator. So we can write

$$\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \dots + \frac{271}{2} = \frac{1 + 3 + 5 + \dots + 271}{2}$$

Now we just have to evaluate the sum of the odd natural numbers from 1 to 271 and then divide the answer by 2.

In order to use the formula for the sum of the first n odd natural numbers, we need to find out how many terms are in the sum in the numerator. To do this, we let $2n - 1 = 271$ and solve for n .

$$2n - 1 = 271$$

$$2n = 271 + 1$$

$$2n = 272$$

$$n = 136$$

Therefore, then we can use the formula we derived in class

$$1 + 3 + 5 + \dots + (2n - 1) = n^2$$

with $n = 136$.

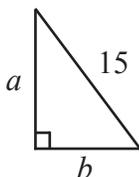
Thus, we have that

$$\begin{aligned} \frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \dots + \frac{271}{2} &= \frac{1 + 3 + 5 + \dots + 271}{2} \\ &= \frac{136^2}{2} \\ &= \frac{18496}{2} \\ &= 9248 \end{aligned}$$

4. If the sum of the lengths of the legs of a right-angled triangle is 21 cm, and if the hypotenuse is 15 cm, then what are the lengths of the triangles legs (answer in cm)? **Hint:** Use Pythagorean triples. There is only one possible set of side lengths.

Solution:

In a question like this, we always should draw a picture! (note that orientation of the triangle doesn't matter and lengths are in cm)



Recall that the legs of a right-angled triangle are the two sides adjacent to the right angle. That means we need to solve for a and b given the information we have.

We are given that the length of the legs of the triangle add up to 21 cm. That is,

$$a + b = 21 \tag{1}$$

Furthermore, because the triangle is right-angled, we could apply Pythagorean Theorem. But this would just give us that

$$a^2 + b^2 = 15^2 = 225 \tag{2}$$

Then we would have a system of two equations with two unknowns. But unlike the systems you have seen so far, this system requires us to solve a *quadratic* equation. You will learn how

to do this in Grade 10.

Instead of this approach, we can use our knowledge of similar triangles and Pythagorean triples to help us.

Notice that $15 \div 5 = 3$, so if we assume that the triangle in the problem is similar to the 3, 4, 5 right-angled triangle, then it must be scaled by a factor of 3. This would tell us that $a = (3)(4) = 12$ and $b = (3)(3) = 9$.

Do these side lengths satisfy equation (1)? YES! Clearly, $12 + 9 = 21$, and so $a = 12$ and $b = 9$ is a valid solution to the problem.

The hint tells us that there is only one possible set of side lengths that satisfies the conditions of the problem, so we have found the only correct answer.

Therefore, the lengths of the legs of the given triangle are 12 cm and 9 cm.

5. Julia delivered newspapers last year (for 52 weeks). The first week, she earned \$30 for the week. Every week, Julia was given a \$1 raise on her weekly salary. How much money did Julia make over the entire year?

Solution:

Week #	1	2	3	4	...	50	51	52
Payment that Week	\$30	\$31	\$32	\$33	...	\$79	\$80	\$81

We need to find the sum of all of the columns in the second row to determine her total pay. We can do this in two ways.

- A. Find the sum from 1 to 81, then subtract the sum from 1 to 29
- B. Use the general formula of an arithmetic sequence

Method A:

$$\begin{aligned} 30 + 31 + 32 + 33 + \dots + 81 &= (1 + 2 + 3 + \dots + 81) - (1 + 2 + 3 + \dots + 29) \\ &= \frac{(81)(82)}{2} - \frac{(29)(30)}{2} \\ &= \frac{6642}{2} - \frac{870}{2} \\ &= \frac{6642 - 870}{2} \\ &= \frac{5772}{2} \\ &= 2886 \end{aligned}$$

Method B:

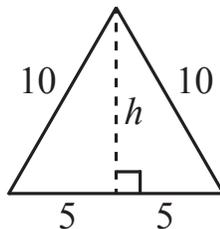
$$\begin{aligned} S &= [(\text{First Term} + \text{Last Term}) \times (\# \text{ of Terms})] \div 2 \\ &= [(30 + 81) \times (52)] \div 2 \\ &= (111 \times 52) \div 2 \\ &= 2886 \end{aligned}$$

Therefore, Julia makes a total of \$2886 over the year.

6. For this question round your answers to the nearest tenth of a cm^2 .
- (a) Find the area of an equilateral triangle with 10 cm sides.

Solution:

Once again, we start by visualizing the question with a picture:



As seen in the diagram above, we can always bisect an equilateral triangle into two congruent right-angled triangles. Therefore, we can use the Pythagorean Theorem to find the height, h , of the equilateral triangle:

$$\begin{aligned} h^2 + 5^2 &= 10^2 \\ h^2 + 25 &= 100 \\ h^2 &= 100 - 25 \\ h^2 &= 75 \\ h &= \sqrt{75} \quad (\text{take positive root}) \\ h &\approx 8.66 \end{aligned}$$

Now, we can just use the formula for the area of a triangle with base $b = 10$ cm and height $h \approx 8.66$ cm

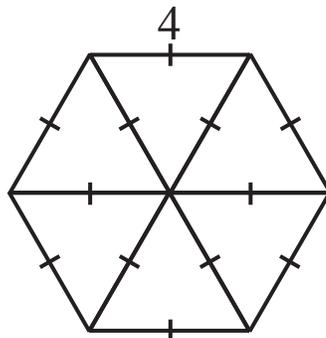
$$\begin{aligned} \text{Area} &= \frac{1}{2}bh \\ &\approx \frac{1}{2}(10)(8.66) \\ &\approx 43.3 \end{aligned}$$

Therefore, the area of an equilateral triangle with 10 cm sides is approximately 43.3 cm^2 .

- (b) Find the area of a regular hexagon with 4 cm sides. **Note:** A regular polygon has all sides equal and all interior angles equal.

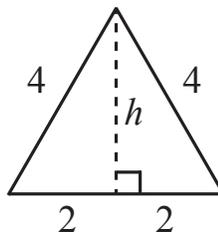
Solution:

The key idea in this question is that we can cut the regular hexagon into six congruent equilateral triangles as shown below: (picture!)



Then we can find the area of one of those triangles in the same way we did in part (a), and then multiply the answer by 6 to get the area of the entire hexagon.

Each equilateral triangle has sides of 4 cm length. They can be bisected into two congruent right-angled triangles as shown below:



Therefore, we can use the Pythagorean Theorem to find the height, h , of the equilateral triangle:

$$h^2 + 2^2 = 4^2$$

$$h^2 + 4 = 16$$

$$h^2 = 16 - 4$$

$$h^2 = 12$$

$$h = \sqrt{12} \quad (\text{take positive root})$$

$$h \approx 3.46$$

Now, we can just use the formula for the area of a triangle with base $b = 4$ cm and height $h \approx 3.46$ cm

$$\begin{aligned} \text{Area} &= \frac{1}{2}bh \\ &\approx \frac{1}{2}(4)(3.46) \\ &\approx 6.93 \end{aligned}$$

Therefore, one of the equilateral triangles with 4 cm sides has an area of approximately 6.93 cm².

Since there are six of those triangles in the regular hexagon with 4 cm sides, the hexagon must have an area of approximately $(6)(6.93) = 41.6$ cm².

7. The world has been taken over by aliens with extremely long life spans (like 100's of years) and fast breeding rates! Every night at midnight, each breeding alien spawns three new aliens. And once an alien is spawned, it will breed the next day! Thankfully, each alien can only breed once in its lifetime. An initial population of 100 breeding aliens landed on planet Earth on Thursday. (Assume that no other aliens land on Earth afterwards)

- (a) If today is Tuesday, what is the current alien population on Earth?

Solution:

The number of breeding aliens on any given day is three times as many as on the previous day. This is because an alien is a breeding alien on any given day if it was spawned the night before (an alien breeds the day after it is spawned and can only breed once), and because each breeding alien spawns 3 new aliens at midnight.

Day	Thurs	Fri	Sat	Sun	Mon	Tues
# of Breeding Aliens	100	300	900	2700	8100	24300

Because the aliens have extremely long life spans, even though an alien has stopped breeding, we must still count it as part of the alien population on Earth. Therefore, since today is Tuesday, the current alien population on Earth is given by the sum of all the columns in the second-row in the table above.

So the current alien population on Earth is 12,100.

- (b) If the current growth rate stays the same, then what will be the alien population on Earth in 2 weeks from today (Tuesday)? **Hint:** There is a formula for this!

Solution:

We could just make a chart like we did in part (a) and add all the terms up, but this would take a really long time!

Instead, observe that the sum in part (a) was a geometric series:

$$100 + 300 + 900 + 2700 + 8100 + 24300$$

The series has initial value $a = 100$, common ratio $r = 3$, and a finite number of terms $n = 6$.

Because the breeding pattern of the aliens continues in the same way, the alien population on Earth two weeks from today is given by a geometric series as well, with $r = 3$, $a = 24300$, and $n = 14$. Since this geometric series is finite, we can use our formula to evaluate it. Let S be the alien population on Earth two weeks from today. Then,

$$\begin{aligned} S &= a \left(\frac{1 - r^n}{1 - r} \right) \\ &= (24300) \left(\frac{1 - 3^{14}}{1 - 3} \right) \\ &= (24300) \left(\frac{-4782968}{-2} \right) \\ &= (24300)(2391484) \\ &= 58113073300 \end{aligned}$$

So the alien population on Earth two weeks from today will be 58 113 073 300. That's more than eight times the current human population!

8. * **Arithmetic Series.**

An arithmetic series is a sum where the terms in the sum are related by a **common difference**. For example, $3 + 7 + 11 + 15 + 19$ is an arithmetic series with a common difference of $d = 4$.

In general, we can define a finite arithmetic series with a common difference d , a starting value (first term) a , and a number of terms n . That is,

$$a + (a + d) + (a + 2d) + \dots + (a + (n - 1)d)$$

is an arithmetic series of n terms, starting at a , with a common difference of d .

- (a) Using the same algebraic strategy that we used in the lesson to derive the formula for the sum of the first n natural numbers, try to derive the formula for the sum of a general arithmetic series:

$$S = \left(\frac{n}{2} \right) (2a + (n - 1)d)$$

where S just represents the value of the sum. **Hint:** You have seen this formula before. It's just written differently now.

Solution:

Start with the general arithmetic series of n terms,

$$S = a + (a + d) + (a + 2d) + \dots + (a + (n - 1)d)$$

Just like we did in the lesson, we can write out the series twice; once in the proper order, and then again in reverse order. Then we line them up so that we can sum term by term. As we saw in the lesson, we will find that $2S$ is the sum of n pairs of the first term of the series plus the last term of the series. That is, we find that

$$2S = (\text{1st Term} + \text{Last Term}) \times (\# \text{ of Terms})$$

Dividing by 2 gives us that

$$S = [(\text{1st Term} + \text{Last Term}) \times (\# \text{ of Terms})] \div 2$$

But notice that the first term of the general arithmetic sequence S is a , and the last term is $(a + (n - 1)d)$. And as was already mentioned, there are n terms in S . Substituting this into the equation above, we get that

$$\begin{aligned} S &= [(a + (a + (n - 1)d)(n)] \div 2 \\ &= \left(\frac{1}{2}\right) [(a + a + (n - 1)d)(n)] \\ &= \left(\frac{n}{2}\right) (2a + (n - 1)d) \end{aligned}$$

as required.

- (b) Use the formula given in part (a) to evaluate the sum of the first 10 terms of the infinite arithmetic series $5 + 13 + 21 + 29 + \dots$

Solution:

To use the formula, we need to know a , d , and n . Since we are finding the sum of the first ten terms, we know that $n = 10$, and we are given that $a = 5$. To find d , we simply subtract any given term by the previous term. That is, $d = 13 - 5 = 8$.

Substituting these values into the formula given in part (a), we get that the sum of the

first 10 terms of the arithmetic series $5 + 13 + 21 + 29 + \dots$ is

$$\begin{aligned} S &= \left(\frac{10}{2}\right) (2(5) + (10 - 1)(8)) \\ &= (5)(10 + (9)(8)) \\ &= (5)(10 + 72) \\ &= (5)(82) \\ &= 410 \end{aligned}$$

9. * An important algebra fact that you will learn in high school is that if you have two whole numbers a and b , then $(a + b)^2 = a^2 + 2ab + b^2$. Complete the algebraic proof and provide a geometric proof of this result using the diagram.

Solution:

(ALGEBRAIC) PROOF:

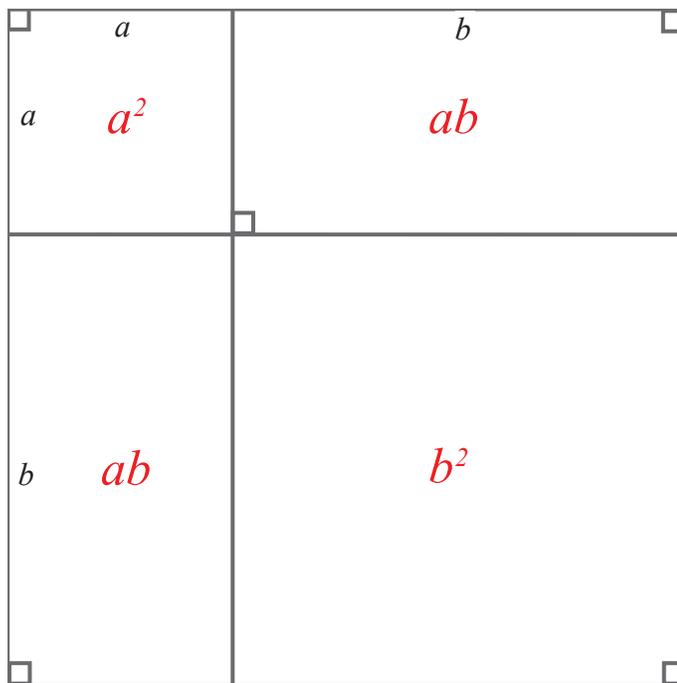
$$(a + b)^2 = (a + b)(a + b)$$

Let $x = (a + b)$. Then

$$\begin{aligned} (a + b)^2 &= (a + b)(a + b) \\ &= x(a + b) \\ &= (x)(a) + (x)(b) && \text{(Distributive Property)} \\ &= (a + b)(a) + (a + b)(b) && \text{(Substitute } x = (a + b)\text{)} \\ &= [a \times a + b \times a] + [a \times b + b \times b] && \text{(Distributive Property again)} \\ &= a^2 + ba + ab + b^2 \\ &= a^2 + 2ab + b^2 && (ab = ba) \end{aligned}$$

□

(GEOMETRIC) PROOF:



The large square with side lengths $(a + b)$ has an area of

$$\text{length} \times \text{width} = (a + b)(a + b) = (a + b)^2$$

It is divided into a square of side lengths a , a square of side lengths b , and a two identical rectangles with side lengths a and b . The areas of these shapes are a^2 , b^2 , and $ab = ba$ respectively.

Since the area of the large square is the same as the sum of the areas of the smaller squares and rectangles, we have that

$$\begin{aligned}(a + b)^2 &= a^2 + b^2 + ab + ab \\ &= a^2 + 2ab + b^2\end{aligned}$$

as required.

□

10. ** Solve for x given that

$$25 = \sqrt{x + \sqrt{x + \sqrt{x + \dots}}}$$

Hint: Use a similar strategy to the algebraic one we used to evaluate the infinite geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

Solution:

Notice that we have an infinite pattern of square roots on the right hand side of the equation. Let's call the pattern P .

That is, let $P = \sqrt{x + \sqrt{x + \sqrt{x + \dots}}} = 25$

Then observe that $P = \sqrt{x + \sqrt{x + \sqrt{x + \dots}}} = \sqrt{x + P}$

Therefore,

$$\begin{aligned} 25 &= \sqrt{x + P} \\ &= \sqrt{x + 25} \end{aligned}$$

Squaring both sides of this equation gives

$$625 = x + 25$$

Hence, it follows that

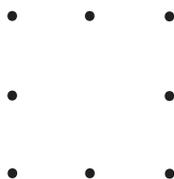
$$x = 600$$

11. Find the sum of the integers from 5 to 34.

Solution:

$$\begin{aligned} S &= (\text{Sum from 1 to 34}) - (\text{Sum from 1 to 4}) \\ &= \frac{(34)(35)}{2} - \frac{(4)(5)}{2} \\ &= 595 - 10 \\ &= 585 \end{aligned}$$

12. * In the following diagram, how many unique ways can you join two points?

**Solution:**

This is essentially the handshake problem covered in the lesson. We have eight points, so we can think of them as eight people. Each time we join a point uniquely, we are doing a unique handshake.

Then the total number of unique ways to join two points is just

$$1 + 2 + 3 + 4 + 5 + 6 + 7 = \frac{(7)(8)}{2} = 28$$

13. Find the sum of this sequence: 1, 4, 7, 10, 11, ..., 73

Solution:

Recall the formula for the sum of an arithmetic sequence:

$$S = [(First\ Term + Last\ Term) \times (\# \text{ of Terms})] \div 2$$

This problem requires us to find the **number** of terms in the sequence, as we know the first term and the last term.

Use some logical thinking. The first term is 1. The last term is 73, and the terms in the sequence increases by 3 each time. The difference between 1 and 73 is $73 - 1 = 72$.

For 1 to get to 73, it has to increase a total of 72 (the difference), but the terms in the sequence increase by 3 each time. Hence 1 has to increase by 3 a total of $72 \div 3 = 24$ times to reach 73. This means that there are 24 terms after the first term, with the 24th term being 73. This means the total number of terms is 25 (24 terms **after** the first term, plus the first term).

Then simply substituting this into the formula gives

$$\begin{aligned} S &= [(First\ Term + Last\ Term) \times (\# \text{ of Terms})] \div 2 \\ &= [(1 + 73) \times (25)] \div 2 \\ &= (74 \times 25) \div 2 \\ &= 1925 \end{aligned}$$

14. Hector and Achilles are having a race. After 1 s, the distance between the two of them increases by 1 cm. After 2 s, the distance between them increases by 4 cm; after 3 s, 7 cm; after 4 s, 10 cm, and so on, with the distance increasing by 3 cm every second. How far apart are they after 25 s?



Solution:

In the first second, the distance between them is 1 (cm). After two seconds, it increases by 4, so the total is $1 + 4 = 5$. After three seconds, it increases by 7, so the total distance is $1 + 4 + 7 = 12$. So the increasing distance between them forms an arithmetic sequence, and to find the total distance, we need to sum together all of the terms in the sequence.

To find the total distance after 25s, we need to first find how much they increase the distance between them at 25 s. The sequence is 1, 4, 7, 10, 13, 16, ..., ?, where ? is the 25th term. However, in the previous problem (question 13), we noticed that sequence (which starts with the same first term and increases by the same amount each time) has 25 terms, and ended in 73. So the 25th term in this sequence is 73. Thus, at 25 s, the distance between the two would be increased by 73 cm.

The sequence is therefore 1, 4, 7, 10, 13, ..., 73. The **total** distance between them is just the sum of these terms. But we already found this in question 13. So the total distance between them is 1924 cm, or 19.24 m.

15. (a) In an arithmetic sequence, the first term is 13 and the terms go up by 5. Find the last term if there are 21 terms in total.

Solution:

$$\text{Last Term: } 13 + (20)(5) = 113$$

- (b) In an arithmetic sequence, the first term is 4 and the terms go up by 3. If the last term in the sequence is 49, how many terms are in the sequence?

Solution:

First Term: 4

Last Term: 49

The difference between these is 45, so 4 must have increased by 3 a total of $\frac{45}{3} = 15$ times to reach 49. Thus the total number of terms is $15 + 1 = 16$.

- (c) In an arithmetic sequence, the last term is -3, and the terms decrease by 5 each time. Find the first term if there are 20 terms in total.

Solution:

Work backwards. If the **last** term is -3, and the terms decrease by 5 each time, if we think of the sequence in reverse (call it the reverse sequence), the first term would be -3, and the sequence **increases** by 5 each time. If there are 20 terms, then, -3 would increase by 5 a total of 19 times, so the last term in the reverse sequence would be $-3 + (5)(19) = -3 + 95 = 92$. But this would be the first term of the original sequence. Hence the first term is 92.

16. Sebastian just had knee surgery. His trainer tells him to resume jogging slowly: 12 minutes the first week, 18 minutes the second week, and so on, increasing the number of minutes by 6 every week. How long will it be before Sebastian is jogging 78 minutes a week?

Solution.

The sequence is 12, 18, 24, ..., 78. The terms increase by 6 each time. The question is asking which term in the sequence 78 is; equivalently, it is asking how many terms are in the sequence. 12 increases by 66 to reach 78. But $66 = 11 \times 6$. So then 78 is the 11th term after 12 (the first term). Hence there are 12 terms in the sequence, and thus Sebastian will be jogging 78 minutes a week by the twelfth week after his surgery.

17. The sum of the interior angles of a triangle is 180° ; a square 360° ; a pentagon 540° . Assuming this pattern continues, what is the sum of the interior angles of a polygon with 14 sides?

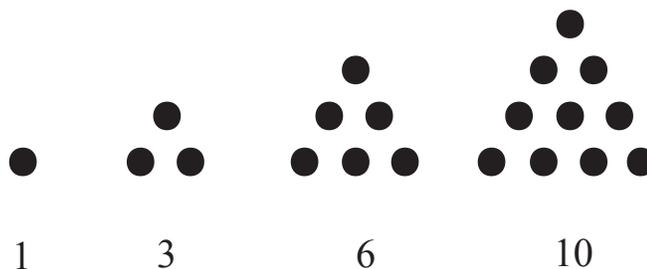
Solution.

The sequence is 180, 360, 540, ... , ?; the terms increase by 180. We have to be a bit careful with this question. The first term corresponds to a triangle, which has **3** sides. So a polygon with 14 sides wouldn't be the 14th term in the sequence, but rather, it would be the 12th term in the sequence. (1st term - 3 sides, 2nd term - 4 sides, ..., n^{th} term - $(n + 2)$ sides). So there are 12 terms in the sequence, up to a polygon with 14 sides.

So the last term in this sequence is $180 + (11)(180) = 2160$. Hence a polygon with 14 sides has the sum of its interior angles as 2160° .

An alternative approach would be to notice that a triangle has $(3 - 2)(180) = 180$, a square has $(4 - 2)(180) = 360$, a pentagon $(5 - 2)(180) = 540$. So a 14 sided polygon would have $(14 - 2)(180) = 12(180) = 2160$ degrees as the sum of its interior angles.

18. The sequence of triangular numbers counts the number of objects that can be used to form an equilateral triangle. (The first 4 are shown below)



What is the 23rd triangular number?

Solution.

The key is to notice that for the n^{th} triangular number, the number of dots is equal to the sum from 1 to n . For example,

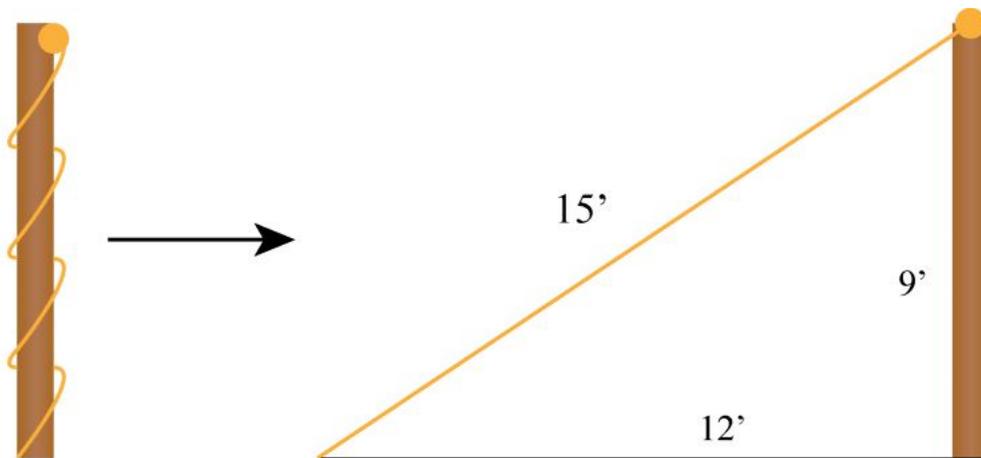
- The first triangular number is 1, which is 1.
- The second triangular number is 3, which is $\underbrace{1 + 2}_{2 \text{ terms}}$
- The third is 6, which is $\underbrace{1 + 2 + 3}_{3 \text{ terms}}$
- The n^{th} triangular number is $\underbrace{1 + 2 + 3 + \cdots + n}_{n \text{ terms}}$

So the 23rd triangular number is the sum of from 1 to 23, which is $\frac{(23)(24)}{2} = 276$.

19. * A python starts at the base of a smooth, cylindrical tree trunk and winds itself tightly 4 times around the trunk until its head rests at the top. The tree trunk has a circumference of 3 feet, and is 9 feet tall. How long is the python?

Solution

Imagine pinning one end of the python to the ground, then unrolling the tree trunk 4 times until the python is stretched out. The total distance moved by the tree trunk as it is unrolled will be $3 \times 4 = 12$ feet, since each time the tree makes a complete revolution, it will have travelled a distance equal to its circumference. This gives the diagram below.



We can assume that the tree trunk is perpendicular to the ground (ie. right angle).

Therefore, since the tree is 9 feet tall, by the Pythagorean Theorem, the length of the python will be the hypotenuse of the triangle, which is $\sqrt{12^2 + 9^2} = 15$ feet.

20. * The 7th term in an arithmetic sequence is 23, and the 12th term is 38. Find the first term and the common difference.

Solution.

The difference between the 7th and 12th term is 15. There are 5 terms after the 7th until the 12th. So the sequence must be increasing by $\frac{15}{5} = 3$, which is the common difference.

The 7th term is 23, so working backwards (by counting backwards by 3), we see that the first term is six terms behind the 7th, and is therefore $23 - (3)(6) = 5$.

21. ** The numbers $x + 1$, $3x - 1$, and $4x + 1$ are consecutive terms in an arithmetic sequence. Find x .

Solution.

These are consecutive terms in an arithmetic sequence. In an arithmetic sequence, the difference between consecutive terms has to be constant.

In particular, the difference between $3x - 1$ and $x + 1$ has to equal the difference between $4x + 1$ and $3x - 1$.

So $4x + 1 - (3x - 1) = 3x - 1 - (x + 1)$ must be true. Simplify both sides of the equation:

$$4x + 1 - (3x - 1) = 3x - 1 - (x + 1)$$

$$4x + 1 - 3x + 1 = 3x - 1 - x - 1$$

$$x + 2 = 2x - 2$$

$$x + 2 - x = 2x - 2 - x$$

$$2 = x - 2$$

$$2 + 2 = x - 2 + 2$$

$$4 = x$$

Therefore $x = 4$.