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Grade 7 & 8 Math Circles
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Mathematical Games

The Loonie Game

In the loonie game, two players, A and B, lay down 17 loonies on a table.

They take turns removing the loonies. They must take at least one loonie, but no more than 3 loonies, at a time. The winner is the one who takes the last loonie (and they get the rest of the loonies for winning). Person A goes first, then Person B.

Turn	# of Loonies Remaining
start	17
A	
B	
A	
B	
A	
B	
A	
B	
A	
B	

Turn	# of Loonies Remaining
start	17
A	
B	
A	
B	
A	
B	
A	
B	
A	
B	

Analysis of the Loonie Game

A **round** of this game occurs whenever Player A finishes his or her move or when the game is over.

If at the *beginning* of a round, Player B is faced with 4 loonies, can he or she win the game? Why?
No, because if Player B takes x loonies (x is 1, 2, or 3), Player A takes $4 - x$ loonies.

Is there a way for Player A to *always* force Player B into the above situation?
Yes if Player A goes first.

What amount of loonies can be removed in one round, no matter how many loonies Player B takes? If Player B takes x loonies (x is 1, 2, or 3), Player A can take $4 - x$ loonies and make sure that 4 loonies are removed in every round.

The Secret to Winning:

1. Player A removes 1 loonie
2. Player B removes x loonies (x is either 1, 2, or 3)
3. Player A removes $4 - x$ loonies (either 1, 2, or 3)
4. Repeat until it is Player B's turn and there are only 4 loonies left
5. Player A is guaranteed to win

Terminology:

In mathematical games, a **strategy** is the collection of *all* the moves made by a player during the course of one game.

A **winning strategy** is a strategy which guarantees a win, *no matter what* the other player(s) do.

A **winning position** is any point in a winning strategy.

The 100 Game

In the 100 Game, two players start with a sum of 0. Alternating turns, they add a number from 1 to 10 to the sum. The first player to reach 100 wins.

Winning Strategy: On the problem set.

Winning Strategies

Do all mathematical games have a winning strategy? No. Example: Tic-tac-toe

Interesting Fact: *If a mathematical game between two players **cannot** end in a tie, then there must be a winning strategy for one of the players.*

PROOF:

Let's call the two players A and B.

- If A has a winning strategy, we're done.
- If A does not have a winning strategy, this means that B must have a way to prevent A from winning.

- So then either both of them tie, or B wins (since A does not win).
- But there are no ties. So B wins.
- The moves that B makes are his/her winning strategy.

Either way, there is a winning strategy!

$\square$

Finding Winning Strategies

We can apply some problem solving techniques to help us.

- Working Backwards
- Symmetry
- Reducing the Problem
- Game Trees

Working Backwards - Example: Nim

In the two-player game of Nim, 33 matchsticks are laid out in a row across a table. On each turn, a player must remove 1, 2, 3, or 4 matchsticks. Whoever clears the table wins the game. Can we have a tie? If not, which player has a winning strategy?

Turn	# of Matchsticks Remaining
start	33
A	
B	
A	
B	
A	
B	
A	
B	
A	
B	
A	
B	
A	
B	

Turn	# of Matchsticks Remaining
start	33
A	
B	
A	
B	
A	
B	
A	
B	
A	
B	
A	
B	
A	
B	

If we work backwards, we see that whoever has to remove from a pile of 5 matchsticks will lose the game. This is known as a **losing position**.

What are the other possible losing positions? All multiples of 5 less than 33

If the person who goes first is A, and the person who goes second is B, can A control the game to always force B into a losing position? Yes, by taking away 3 matchsticks on his/her first move.

Symmetry - Example: Nim 2.0

In certain games involving two players, one player can often force a winning position by matching or mirroring their opponent's moves. This is known as the **tit-for-tat** or **symmetry** strategy.

In this variation of Nim, there are now two piles of matchsticks on the table, one with 15 matches, another with 19 matches. Each player is allowed to take 1, 2, 3, or 4 matches from either pile, but cannot take from both at once. Once again, the player who clears the last matchstick off the table wins. Can we have a tie? Which player has a winning strategy?

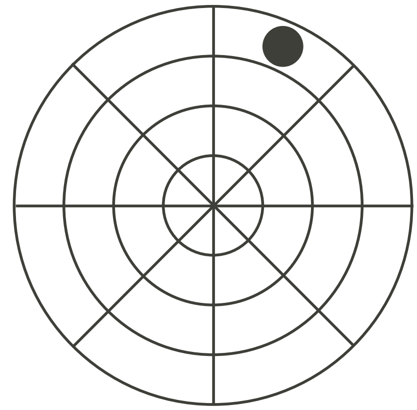
Turn	Pile 1	Pile 2
start	19	15
A		
B		
A		
B		
A		
B		
A		
B		
A		
B		
A		
B		
A		
B		

Turn	Pile 1	Pile 2
start	19	15
A		
B		
A		
B		
A		
B		
A		
B		
A		
B		
A		
B		
A		
B		

The secret: match the piles.

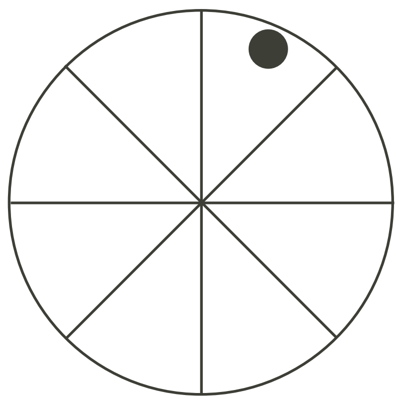
The person who is faced with an equal number of matchsticks in both piles at the beginning of their turn *cannot* use a winning strategy. Since there are no ties, the other player has a winning strategy.

Reducing the Problem - Example: Spiral Game



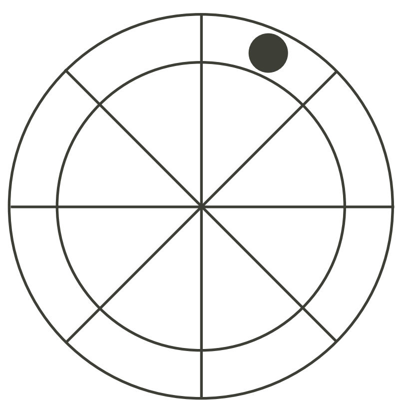
- Two players, one piece, circular board
- Can move the piece clockwise one position, or one position towards centre
- Cannot move counterclockwise or outwards, or onto a position that has already been occupied
- Winner is the person who makes the last move

There are 4 rings in the board diagram. Let's break the problem down and consider only the innermost ring, like below.



Key Point: Whoever makes the first move in the innermost ring always wins the game!

But who reaches the innermost ring first? Let's consider the two inner rings.



Key Point: Whoever moves first in an outer ring can also move first in the next inner ring!

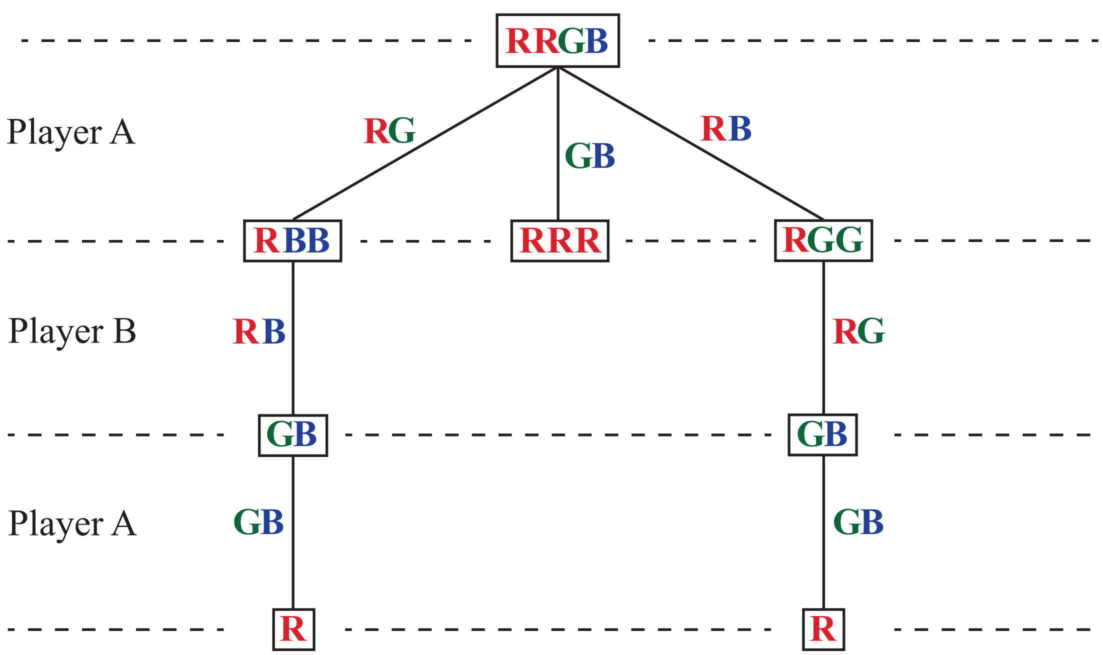
So the winning strategy is: The person who goes first moves one space clockwise each turn.

Game Trees - Example: The Replacement Game

In the Replacement Game, there is a jar filled with 2 red marbles, 1 green marble, and 1 blue marble. Two players, A and B, take turns removing marbles from the jar. On each turn, a player **must** remove two marbles of differing types from the jar, and replace it with a marble of the other type that they did not take. For example, if A removed a red and a green marble, he/she must return a blue marble. The winner is the player who makes the last possible legal move.

What are the possible winning strategies, if any? Who has them - Player A (goes first), or Player B (goes second)?

Let R represent one red marble, G represent one green marble, and B represent one blue marble. Then the original set-up in the jar is RRGB. We draw a game tree by beginning at this point and creating branches which represent possible moves. At the end of each branch, the result of that move is recorded. We label the turns of each player. Each path that can be traced leads to a possible end-result of the game.



All possible paths end with a move by player A. Therefore, Player A will always win the game. So the winning "strategies" (they don't require much thinking) include all possible collections of moves by Player A.

This game is pretty boring, but it was just used so that you could learn about game trees. Game trees can be useful for studying more interesting games as well. In fact, game trees can tell us whether winning strategies exist, and if so, what they are, for any mathematical game. The downside to game trees is that they take a long time to make for long games.

Problem Set

*difficult **challenge ***mad skillz

For the following problems, it is recommended you actually try playing the games with a partner to gain a better understanding. Remember the problem-solving strategies we covered in the lesson! Explain your answers!!

1. **Classic Nim and Variations**

   (a) If the Nim Game started with 40 matchsticks, and you could remove 1, 2, 3, or 4 matchsticks, which player has the winning strategy?

   (b) If the Nim Game started with 37 matchsticks, and you could remove 1, 2, or 3 matchsticks, which player has the winning strategy?

   (c) In *Misere Nim*, the **loser** is the one who **takes the last matchstick**.

      i. If the Nim Game started with 33 matchsticks, and you could remove 1, 2, 3, or 4 matchsticks, which player has the winning strategy?

      ii. If the Nim Game started with 36 matchsticks, and you could remove 1, 2 or 3 matchsticks, which player has the winning strategy?

2. **Nim 2.0 and Variations**

   (a) If one pile had 21 matches and the other had 23, and you could take 1, 2, 3, or 4 from either pile per turn, which player has the winning strategy?

   (b) * If the piles have 21 and 26 matches respectively, and you can take 1 to 4 matches from either pile, who has the winning strategy?

3. Jack and the Giant are playing a game with two piles of beans. On each player's turn, they must remove at least 1, but at most 7 beans from one of the piles. The winner is the player to take the last bean.

   (a) It is Jack's turn and the two piles have 25 and 27 beans. What is Jack's best move?

   (b) ** If the two are playing *Misere* style, where the loser is the one who takes the last bean, who has the winning strategy? Assume the same situation as in part (a).

4. * In the 100 Game described in class, explain how the person who goes first always has a winning strategy. (**Hint:** work backwards and figure out the winning positions)

5. * Adam and Amina are playing a game with four stacks of cards on a table. On each player's turn, he/she must remove some cards from any **one** stack (i.e. they can remove anywhere from 1 card to all of the cards in the stack). The four stacks have 11, 11, 14, and 16 cards. The winner is the player who removes the last card from the table. Since ladies go first, explain Amina's strategy.

6. Bill and Steve made up their own mathematical game during math class. The game begins with a rook on the square at the bottom-left corner of a standard 8×8 chessboard. The rules are as follows:

   i. Players take turns moving the rook any number of squares up or to the right.

   ii. The rook cannot be moved down or to the left.

   iii. The rook must be moved at least one square every turn (ie. no passing your turn)

   iv. The player who moves the rook to the square at the top-right corner of the board wins the game.

   Answer the following questions about the game.

   (a) How can you be sure that there is a winning strategy to the game?

   (b) If Bill makes the first move of the game, then which player has a winning strategy? Explain the winning strategy.

   (c) Bill and Steve invite their friend Natasha to play the game with them. Bill suggests that Natasha should go first, then Steve, and then himself. Steve and Natasha agree with the playing order, but Steve also says that they should modify rule i. so that players can only move the rook **one** square up or to the right. He claims that this will help avoid quick games.

   Why do the new rules guarantee that Steve will win the game no matter what?

7. There are 2 apples, 1 orange, and 3 peaches in a basket. Just like the Replacement Game covered in the lesson, two players, A and B, take turns removing two different types of fruit from the basket, then replace it with 1 fruit of the remaining type. The winner is the player who makes the last legal move.

   (a) If B goes first, who wins the game? (**Hint:** Game Tree)

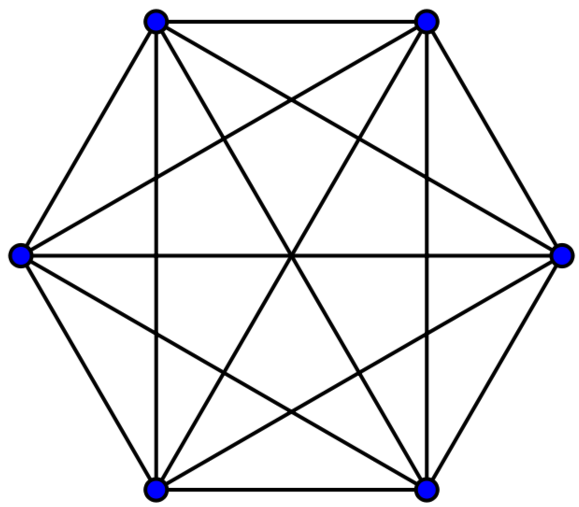
   (b) What fruit is left in the basket?

8. John and Bert are playing a rather strange card game. They start with 7 cards. John starts the game by discarding at least one, but no more than half the cards in the pack (so he can't discard 4, since $4 > 3.5$). He then gives the remaining cards to Bert. Bert continues by discarding at least one, but no more than half the remaining cards. This continues on, with each player taking a turn and passing it back and forth. The loser is the player who is left with the last card.

   (a) Bert has the winning stratey. Explain what it is (**Hint:** think even and odd numbers, and work backwards).

(b) ** After losing, John demands a rematch. This time John starts with 52 cards. The rules stay the same - a player can remove anywhere from 1 to half the cards in the deck, and then passes it on to the other player, who then repeats this. The loser is the one who is left with the last card. Does John have a winning strategy?

9. **The Sim Game**. The Sim Game requires two players, Red and Green. Red goes first. The idea is to color the lines running between the 6 vertices in the graph below, either Red or Green. The first player to colour three lines forming a triangle (between three of the six vertices of the graph) in the same colour wins the game.



(a) Play a couple of rounds - is there a possibility of a tie?

(b) *** To show that there is a winning strategy, we have to show that no matter how we colour the lines, there must be three lines forming a triangle which all have the same colour.

i. In the diagram, pick a vertex and label it O.

ii. There are five lines connected to O. Explain why, no matter how you colour the five lines, that three of these lines have to be the same colour.

iii. Label the vertices that these lines are connected to as A, B, C. Then OA, OB, and OC are the same colour.

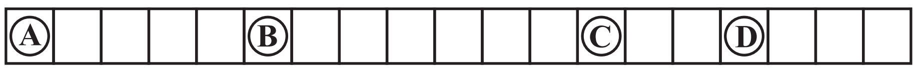
iv. Explain what happens if at least one of AB, BC, or AC is the same colour as OA, OB, and OC. What happens if none of AB, BC, or AC are the same colour as OA, OB, or OC?

v. Explain why there must be a winning strategy.

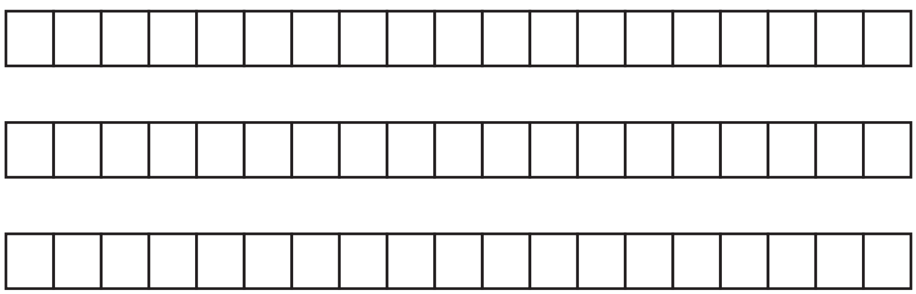
10. ** A game for two players uses four counters on a board which consists of a 20×1 rectangle. The two players alternate turns. A turn consists of moving any one of the four counters any number of squares to the right, but the counter may not land on top of, or move past, any of the other counters. For instance, in the position shown below, the next player could move D one, two or three squares to the right, or move C one or two squares to the right, and so on.

    The winner of the game is the player who makes the last legal move. (After this move the counters will occupy the four squares on the extreme right of the board and no further legal moves will be possible.)

    In the position shown below, it is your turn. Which move should you make and what should be your strategy in later moves to ensure that you will win the game?

    Take turns, placing A, B, C, and D on the following puzzles, see if you can find a winning strategy with a friend.

11. ** **Sprouts**. With a partner, draw one to four dots on a page. Take turns, where each turn consists of drawing a line between two dots (or from a dot to itself) and adding a new dot somewhere along the line. You must follow three rules:

    - The line may be straight or curved, but must not touch or cross itself or any other line
    - The new dot cannot be placed on top of one of the endpoints of the new line. Thus the new spot splits the line into two shorter lines
    - No dot may have more than three lines attached to it. A line from a spot to itself counts as two attached lines and new dots are counted as having two lines already attached to them.

    Whoever makes the last move loses (*Misere* play). Can you determine if there is a winning strategy for 1, 2, 3, and 4 initial dots? [1]

[1] This game is actually played competitively. Notes from `http://en.wikipedia.org/wiki/Sprouts_(game)`