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Grade 7 & 8 Math Circles

November 19/20/21, 2013

Mathematical Games - Solutions

1. **Classic Nim and Variations**

   (a) If the Nim Game started with 40 matchsticks, and you could remove 1, 2, 3, or 4 matchsticks, which player has the winning strategy?

   Solution:

   In Classic Nim, there are always two players, so let Player A go first, Player B go second. For games of Classic Nim, it is often best to work backwards.

   A player faced with 5 matchsticks remaining on his/her turn (ie. at the beginning of the player's turn) is in a *losing position*. This is because the player may only take x matchsticks (where x is 1, 2, 3, or 4), and then the other player will just take $(5 - x)$ matchsticks to win the game.

   In any two consecutive turns of the game, 5 matchsticks can always be removed if desired. Therefore, a player faced with any multiple of 5 matchsticks remaining on his/her turn is in a losing position.

   Player A makes the first move of the game and is faced with 40 matchsticks. Since 40 is a multiple of 5, Player A begins the game in a losing position. On the other hand, Player B is in a *winning position* to begin the game, because he/she can always force Player A into a losing position by taking $(5 - x)$ matchsticks after Player A takes x matchsticks. This is **Player B**'s winning strategy.

   (b) If the Nim Game started with 37 matchsticks, and you could remove 1, 2, or 3 matchsticks, which player has the winning strategy?

   Solution:

   Once again, let Player A go first, Player B go second.

   Using the same logic as in part (a), we can see that a player is in a losing position if he/she is faced with a multiple of 4 matchsticks remaining on his/her turn. Since 37 is not a multiple of 4, Player A begins the game with an advantage. If Player A removes

1 matchstick to begin the game, then Player B will be forced into a losing position (36 is a multiple of 4). In later turns, Player A removes $(4 - x)$ matchsticks after Player B removes x matchsticks (x is 1, 2, or 3). This is **Player A**'s winning strategy.

(c) In *Misere Nim*, the **loser** is the one who **takes the last matchstick**.

i. If the Nim Game started with 33 matchsticks, and you could remove 1, 2, 3, or 4 matchsticks, which player has the winning strategy?

Solution:

Once again, let Player A go first, Player B go second. We will work backwards.

Obviously, the simplest losing position to think of is when a player is faced with just one remaining matchstick on his/her turn. This is because they must take the matchstick (no passing turns allowed) and we are playing *Misere* Nim.

In any two consecutive turns, 5 matchsticks can always be removed if desired. Therefore, a player is in a losing position whenever he/she is faced with 6, 11, 16, 21, 26, or 31 matchsticks (any number divisible by 5 with remainder 1) on his/her turn.

If Player A begins the game by removing 2 matchsticks, then Player B will be faced with 31 matchsticks at the beginning of his/her turn. Thus, Player B will be in a losing position, and Player A will be in a winning position. In later turns, Player A removes $(5 - x)$ matchsticks after Player B removes x matchsticks (x is 1, 2, 3, or 4). This is **Player A**'s winning strategy.

ii. If the Nim Game started with 36 matchsticks, and you could remove 1, 2 or 3 matchsticks, which player has the winning strategy?

Solution:

Let Player A go first, Player B go second.

Once again, a player is in a losing position when he/she is faced with just one remaining matchstick on his/her turn.

Since 4 matchsticks can always be removed in any two consecutive turns, a player is also in a losing position whenever he/she is faced with 5, 9, 13, 17, 21, 25, 29, or 33 matchsticks (any number divisible by 4 with remainder 1) on his/her turn.

If Player A begins the game by removing 3 matchsticks, then Player B will be faced with 33 matchsticks at the beginning of his/her turn. Thus, Player B will be in a losing position, and Player A will be in a winning position. In later turns, Player A removes $(4 - x)$ matchsticks after Player B removes x matchsticks (x is 1, 2, or 3). This is **Player A**'s winning strategy.

2. **Nim 2.0 and Variations**

(a) If one pile had 21 matches and the other had 23, and you could take 1, 2, 3, or 4 from either pile per turn, which player has the winning strategy?

Solution:

In Nim 2.0, there are always two players, so let Player A go first, Player B go second. The key to understanding Nim 2.0 is to use symmetry.

If a player is faced with two piles that have an equal number of matchsticks on his/her turn, then that player is in a losing position. This is because the other player can match him/her tit-for-tat (symmetrically).

If Player A begins the game by removing 2 matchsticks form the pile of 23, then Player B will be faced with two piles of 21 matchsticks on his/her turn. Thus, Player B will be in a losing position, and Player A will be in a winning position. In all later moves, Player A simply matches Player B tit-for-tat until Player B is forced to clear one pile and Player A gets to clear the other. This is **Player A**'s winning strategy.

(b) * If the piles have 21 and 26 matches respectively, and you can take 1 to 4 matches from either pile, who has the winning strategy?

Solution:

Once again, let Player A go first, Player B go second. Then **Player B** has the winning strategy.

Just like in part (a), if a player is faced with two piles that have an equal number of matchsticks on his/her turn, that player is in a losing position (because of symmetry).

If Player A removes x matchsticks from the pile of 26 matchsticks (x is 1, 2, 3, or 4), then Player B can remove $(5 - x)$ matchsticks from the same pile and force Player A into the losing position described above. Player B can then win the game by matching Player A tit-for-tat.

However, if Player A recognizes this and instead picks x matchsticks from the pile of 21 (x is 1, 2, 3, or 4), then Player B should pick $(5 - x)$ matchsticks from the same pile. In later turns, Player A will pick x matchsticks from one of the two piles and Player B should pick $(5 - x)$ matchsticks from the same pile. This will eventually result in both piles having an equal number of matchsticks on Player A's turn, and so Player B will then switch to tit-for-tat strategy, or this will result in Player A clearing one of the two piles while the other (remaining) pile has 26, 21, 16, 11, 6, or 1 matchstick(s). Thus, Player B can simply take one matchstick from the only remaining pile and hence force Player A into a losing position in what has become a game of Classic Nim (multiple of 5 in this case).

3. Jack and the Giant are playing a game with two piles of beans. On each player's turn, they must remove at least 1, but at most 7 beans from one of the piles. The winner is the player to take the last bean.

   (a) It is Jack's turn and the two piles have 25 and 27 beans. What is Jack's best move?

Solution:

Jack and the Giant are playing a Nim 2.0 variation.

Using the same reasoning as in 2(a), Jack's best move is to remove 2 beans from the pile of 27. This will put the giant in a losing position, and Jack can simply match the giant tit-for-tat after that until Jack wins the game.

(b) ** If the two are playing *Misere* style, where the loser is the one who takes the last bean, who has the winning strategy? Assume the same situation as in part (a).

Solution:

Jack still has the winning strategy. Showing this requires a little bit of creative thinking.

To begin, Jack removes 3 beans from the pile of 27 so that the giant is faced with two piles of 25 and 24 beans.

Note that 8 beans can always be removed in any two consecutive turns. Hence, in all later moves, the giant will take x beans from one of the two piles (where x is an integer from 1 to 7) and Jack should take $(8-x)$ beans from the same pile. That is, Jack should treat the two piles as two separate games of 24-matchstick Classic Nim. Then Jack will "win" both games of 24 matchstick Nim, and the giant will be left with the one remaining bean from the pile of 25 beans.

Note that there is one case where Jack cannot perform such a move. If the giant decides to clear the last bean from the pile that initially had 25 beans at some point in the game while the other pile has not yet been cleared (in this case $x = 1$), then Jack's strategy is to take $(8-1=7)$ beans from the remaining pile and continue picking $(8-x)$ beans from that pile in later turns (where x is the number of beans that the giant picks in later turns).

To be more clear, this special case is when it is the giant's turn and the 25 pile is now left with one bean, and the 24 pile now has 24, 16, or 8 beans. If the giant clears the 25 pile by picking the one bean, then Jack ought to pick 7 beans from the pile of 24, 16, or 8 beans. This will ensure that the pile has one more bean than a multiple of 8 on the giant's turn. Thus, the giant will be left with the last bean if Jack continues to pick $(8-x)$ beans per turn.

4. * In the 100 Game described in class, explain how the person who goes first always has a winning strategy. (**Hint:** work backwards and figure out the winning positions)

Solution:

Let Player A go first, Player B go second.

Observe that the 100 Game is essentially just Classic Nim in reverse. It is as if there are 100 matchsticks and the players remove 1 to 10 matchsticks on each of their turns. Getting to 100 is the same as taking the last matchstick.

We can work backwards and identify the winning and losing positions.

A player is in a losing position when faced with a sum of 89 on his/her turn. If the player adds x to the sum (where x is an integer from 1 to 10), then the other player will add $(11 - x)$ and reach 100 first. You can also think of this losing position as a player faced with 11 matchsticks remaining on his/her turn.

In this game, the other losing positions are when a player is faced with a sum of 1, 12, 23, 34, 45, 56, 67, or 78 on his/her turn. These positions correspond to multiples of 11 matchsticks remaining on the player's turn.

If Player A begins the game by adding 1 (or removing one matchstick), then Player B will be in a losing position and hence Player A will be in a winning position. Then, Player B can only add x to the sum (where x is an integer between 1 and 10). Thus, Player A can force Player B into the next losing position by adding $(11 - x)$ to the sum. Player A continues to do this in later turns until he/she reaches 100 first. This is the winning strategy for Player A (the player who goes first).

5. * Adam and Amina are playing a game with four stacks of cards on a table. On each player's turn, he/she must remove some cards from any **one** stack (i.e. they can remove anywhere from 1 card to all of the cards in the stack). The four stacks have 11, 11, 14, and 16 cards. The winner is the player who removes the last card from the table. Since ladies go first, explain Amina's strategy.

   Solution:

   We will use symmetry to identify Amina's winning strategy.

   Since Amina makes the first move, notice that if she removes 2 cards from the stack of 16, then Adam is faced with two stacks of 11 cards and two stacks of 14 cards on his turn. Observe that this set-up is basically two games of Nim 2.0 happening at the same time!

   If Adam removes any number of cards from one of the 11 card stacks, then Amina can match Adam tit-for-tat by removing the same number of cards from the other 11 card stack.

   Similarly, if Adam removes any number of cards from one of the 14 card stacks, Amina can match Adam tit-for-tat by removing the same number of cards from the other 14 card stack.

   As long as Amina treats each pair of stacks as a separate game of Nim 2.0 and matches Adam's moves tit-for-tat, then she will eventually remove the last card from the table and win the game.

   So Amina's winning strategy is to remove 2 cards from the stack of 16 on her first turn, and then match Adam tit-for-tat in all later turns.

6. Bill and Steve made up their own mathematical game during math class. The game begins

with a rook on the square at the bottom-left corner of a standard 8×8 chessboard. The rules are as follows:

i. Players take turns moving the rook any number of squares up or to the right.

ii. The rook cannot be moved down or to the left.

iii. The rook must be moved at least one square every turn (ie. no passing your turn)

iv. The player who moves the rook to the square at the top-right corner of the board wins the game.

Answer the following questions about the game.

(a) How can you be sure that there is a winning strategy to the game?

Solution:

1 The game described in the question is a mathematical game.

2 There is no possibility of a tie in the game (since only one player can move the rook to the top-right corner of the board).

3 There are two players in the game (Bill and Steve).

Therefore, by the result we proved in the lesson, there must be a winning strategy to the game.

(b) If Bill makes the first move of the game, then which player has a winning strategy? Explain the winning strategy.

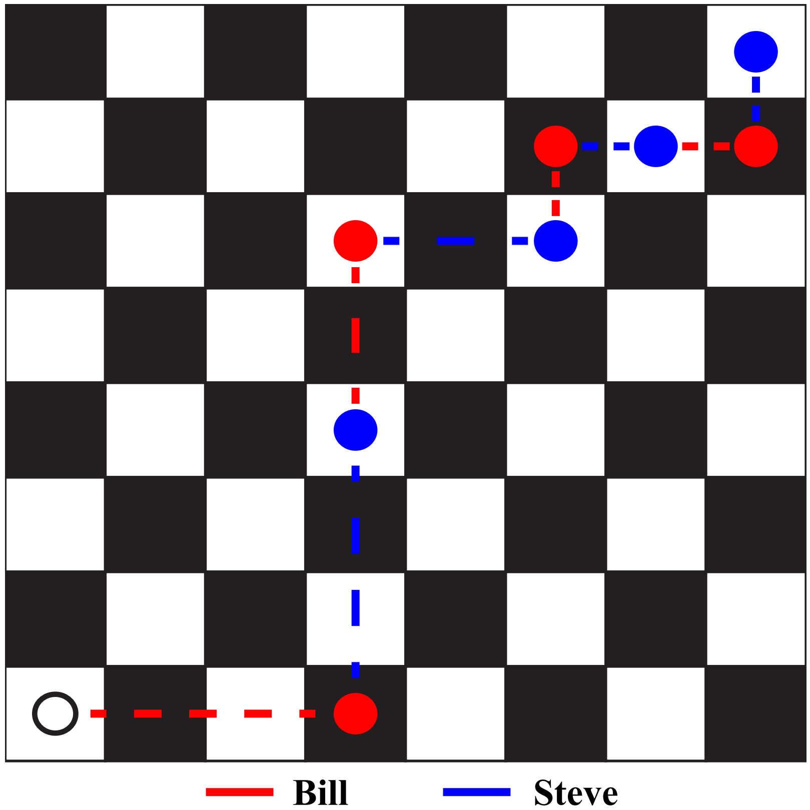
Solution:

If Bill goes first, then Steve has the winning strategy.

Observe that the 8×8 chessboard is perfectly symmetric. Therefore, if Bill moves the rook x squares up (where x is an integer between 1 and 7) and Steve moves the rook x square to the right, the result of the two moves is that the rook has "moved" x squares along the diagonal path between the bottom left-corner of the board and the top-right corner of the board. Similarly, if Bill moves the rook x squares to the right, Steve can move the rook x squares up to achieve the same net result.

As long as the rook is on a square along this diagonal at the beginning of Bill's turn, then Bill *must* move the rook off the diagonal path (since the rook can only move up or to the right). Steve can then move the rook back on to the diagonal until he eventually places the rook in the square at the top-right corner of the board to win. This is Steve's winning strategy.

Below is an example of a possible game between Bill and Steve showing the winning strategy. Notice how all of Steve's moves involve placing the rook on the diagonal line between the bottom-left corner and the top-right corner of the board.



You can also figure this out by working backwards or breaking the game down into smaller games. That is, think about a 2×2 board with the same starting position and rules, then think of a 3×3 board with the same starting position and rules, and so on. You will realize that in any $n \times n$ board (where $n \geq 2$ is a positive integer), the player who goes first begins the game in a losing position, and the player who goes second begins the game in a winning position. The player who goes second can always force the other player into another losing position by using the strategy described above.

(c) Bill and Steve invite their friend Natasha to play the game with them. Bill suggests that Natasha should go first, then Steve, and then himself. Steve and Natasha agree with the playing order, but Steve also says that they should modify rule i. so that players can only move the rook **one** square up or to the right. He claims that this will help avoid quick games.

Why do the new rules guarantee that Steve will win the game no matter what?

Solution:

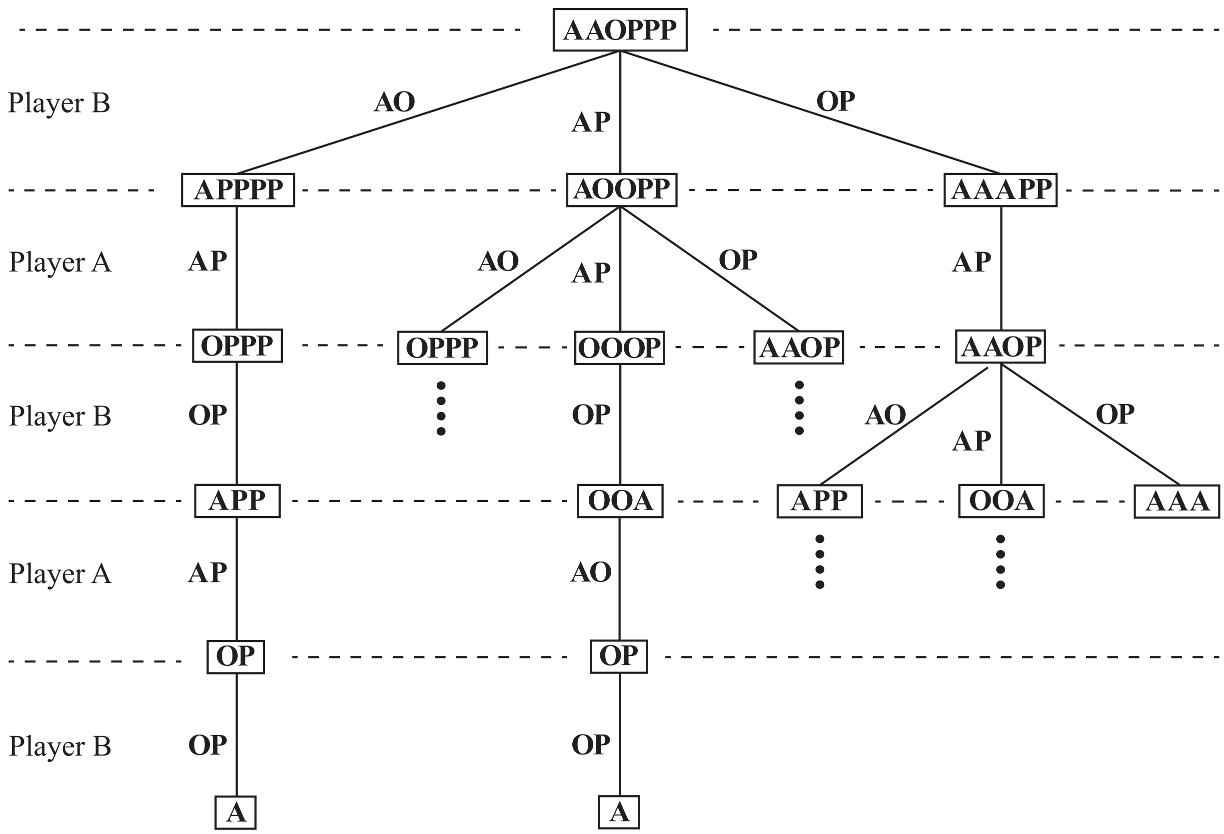
No matter what path the rook takes to get from the bottom left-corner of the board to the top-right corner of the board, the rook must be moved exactly 14 squares (seven to the right, seven up). Since the rook can only be moved 1 square per turn, this means that the game will always take 14 turns to complete. Since $14 \div 3 = 4$ remainder 2, Natasha, Steve, and Bill each have make four moves, and then Natasha makes the 13th move, and Steve makes the 14th move to win. Once again, this result is true for any possible path that the rook takes (ie. any possible set of moves), and therefore Steve will win the game no matter what.

7. There are 2 apples, 1 orange, and 3 peaches in a basket. Just like the Replacement Game covered in the lesson, two players, A and B, take turns removing two different types of fruit from the basket, then replace it with 1 fruit of the remaining type. The winner is the player who makes the last legal move.

   (a) If B goes first, who wins the game? (**Hint:** Game Tree)

   Solution:

   We can construct a game tree for this game as seen below. Note that Player B goes first.



   The vertical dots indicate that a specific branch of the game tree is exactly the same as another branch that has been drawn in full. For example, there are dots below OPPP after the second move of the game. This is because the branch will play out the same way as the far-left branch of the tree. This helps us to save space in large game trees.

   From the game tree, it is clear that no matter what moves are made, Player B will always win the game if he/she goes first.

   (b) What fruit is left in the basket?

   Solution:

   Using the game tree, we see that no matter what moves are made, the game will end with apple(s) in the basket. This is also true even if Player A goes first. In that case, the game tree would look identical to the one above, except that Player A and Player B would change positions in the game tree.

8. John and Bert are playing a rather strange card game. They start with 7 cards. John starts the game by discarding at least one, but no more than half the cards in the pack (so he can't discard 4, since $4 > 3.5$). He then gives the remaining cards to Bert. Bert continues by discarding at least one, but no more than half the remaining cards. This continues on, with each player taking a turn and passing it back and forth. The loser is the player who is left with the last card.

   (a) Bert has the winning stratey. Explain what it is (**Hint:** think even and odd numbers, and work backwards).

   Solution:

   This game is very similar to Classic Nim except for that there is no fixed number of objects that can be removed each turn. We will analyse this game in a similar way to Classic Nim. Let's begin by finding some losing positions.

   The simplest losing position in the game is when a player is faced with one card on his turn. He is left with the last card and therefore loses the game.

   Now observe that a player faced with 3 cards on his turn cannot win the game. This is because the player must discard at least one card from the three but cannot discard more than one card (since $2 > 1.5$). That is, the player must discard exactly one card. Then, his opponent simply discards one card from the remaining two and wins the game. Thus, a player faced with 3 cards on his turn is in a losing position.

   Realizing that John can discard 1, 2 , or 3 cards, we see that Bert will be faced with 4, 5, or 6 cards on his turn. Since half of 4 is 2, half of 5 is 2.5, and half of 6 is 3, Bert can force John into the losing position of 3 cards no matter what move John makes on his first turn. Therefore, 7 is also a losing position. And so Bert's winning strategy is to force John into the losing position of 3 cards. After successfully doing that, Bert is guaranteed to win the game.

   (b) ** After losing, John demands a rematch. This time John starts with 52 cards. The rules stay the same - a player can remove anywhere from 1 to half the cards in the deck, and then passes it on to the other player, who then repeats this. The loser is the one who is left with the last card. Does John have a winning strategy?

   Solution:

   Yes, he does.

   Observe that $(2)(1) + 1 = 3$ and $(2)(3) + 1 = 7$. That is, we can figure out the next two losing positions from the simplest losing position. Does this pattern continue? Is the next losing position $(2)(7) + 1 = 15$, followed by 31?

We can show that a player faced with 15 cards or 31 cards on his turn is also in a losing position as follows.

If John discards 21 cards to begin the game, then Bert will be faced with 31 cards. Bert will then discard x cards (where x is an integer from 1 to 15), and so John can discard $(16 - x)$ cards to ensure that Bert is faced with 15 cards on his next turn. He can do this because $(16 - x)$ is at most 15 (if $x = 1$) which is half of 30 $(31 - x = 31 - 1 = 30)$. John can then make sure that Bert is faced with 7 cards on his next turn and so on.

To gain a better understanding of the problem, we need to understand why $(n+1)$ cards can be discarded in any two consecutive turns if there are $(2n+1)$ cards at the beginning of the first of the two consecutive turns. I have written a brief explanation below for those interested students.

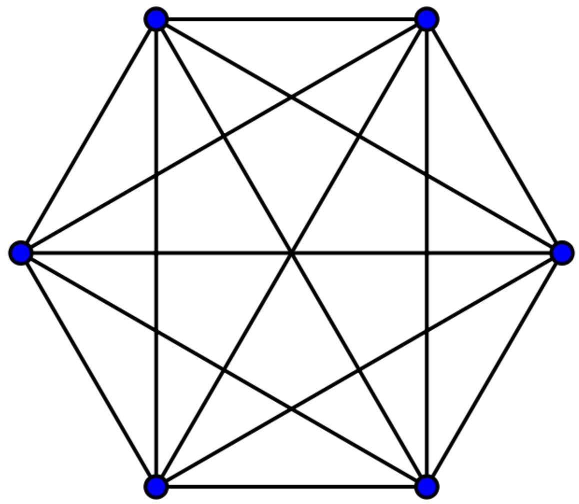
For any given positive integer n, $2n$ is even and $(2n+1)$ is the next positive integer which is odd. So we can denote an even number of cards by $2n$ and an odd number of cards by $(2n+1)$. Note that half of $2n$ is exactly n, but half of $(2n+1)$ is n plus a half. Since we cannot discard half of a card, if a player is faced with a stack of $2n$ or $(2n+1)$ cards on his turn, he many discard at most n cards. Therefore, if a player is faced with $(2n+1)$ cards and he takes one card or n cards, the other player can then take n or one card to guarantee that $(n+1)$ cards are taken in two consecutive turns.

If a player (say Player A) is faced with $(2n+1)$ cards and he takes some integer number of cards between 1 and n to leave the other player (say Player B) faced with $(n+a)$ cards $(1 < a < n)$, then Player B can still remove the a cards to ensure that a total of $(n+1)$ cards are taken in the two consecutive turns (Player A's turn followed by Player B's turn). The reason Player B can always do this is because the average of n and a is $\frac{n+a}{2}$ and is always greater than a since $n > a$. If $(n+a)$ is odd, then half of $(n+a)$ (or the average of n and a) will not be an integer. However, the closest integer less than that value will be greater than or equal to a (if you don't believe me, plug in numbers and see for yourself).

So in any case, if Player A is faced with an odd number of cards on his turn, say $(2n+1)$, then Player B can ensure that exactly $(n+1)$ cards are removed in Player A's turn and his turn combined so that Player A will be faced with n cards on his next turn.

Now we can see why being faced with 15 or 31 cards is also a losing position. If $n = 7$, then $(2n+1) = 15$ and so after removing $(n+1) = 8$ cards in two consecutive turns, the player that was faced with 15 cards initially is now faced with 7 cards, and we know this to be a losing position from part (a). Similarly, If we take this losing position as n, that is let $n = 15$, then the next losing position is found to be $(2n+1) = 31$.

9. **The Sim Game**. The Sim Game requires two players, Red and Green. Red goes first. The idea is to color the lines running between the 6 vertices in the graph below, either Red or Green. The first player to colour three lines forming a triangle (between three of the six vertices of the graph) in the same colour wins the game.



   (a) Play a couple of rounds - is there a possibility of a tie?

   Solution:

   You should not have any ties.

   (b) *** To show that there is a winning strategy, we have to show that no matter how we colour the lines, there must be three lines forming a triangle which all have the same colour.

      i. In the diagram, pick a vertex and label it O.
      ii. There are five lines connected to O. Explain why, no matter how you colour the five lines, that three of these lines have to be the same colour.
      iii. Label the vertices that these lines are connected to as A, B, C. Then OA, OB, and OC are the same colour.
      iv. Explain what happens if at least one of AB, BC, or AC is the same colour as OA, OB, and OC. What happens if none of AB, BC, or AC are the same colour as OA, OB, or OC?
      v. Explain why there must be a winning strategy.

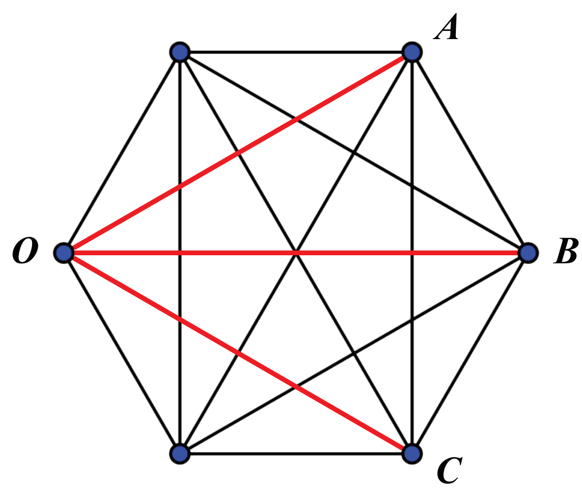
   Solution:

      i. Let the far left vertex be labelled O.
      ii. If you think of the 5 lines as objects that we can place in either a red bin or a green bin (corresponding to the line being coloured red or green respectively), then no

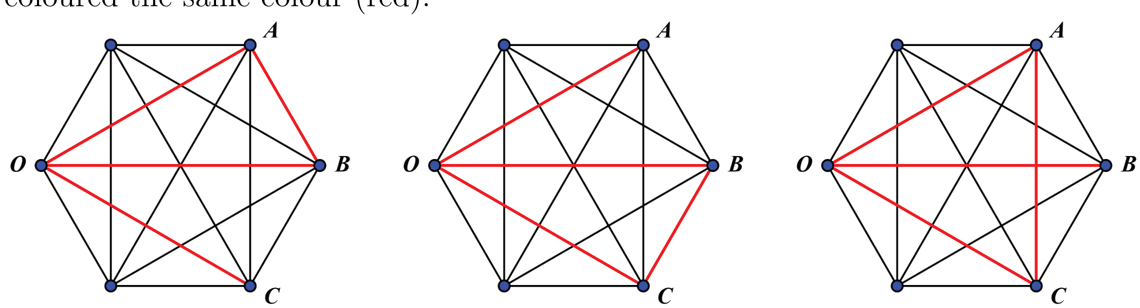
matter how we distribute the objects (lines) in the bins, at least one of the bins will 3 objects (lines) in it. That is, we can put all five objects in one bin, four in one bin and one in the other, or 3 in one bin and 2 in the other.

Note: This idea is called the Pigeon Hole Principle. It is a simple but powerful result that is often used in proofs.

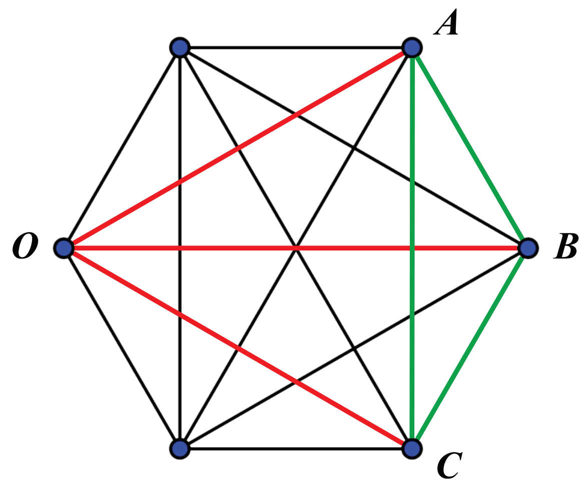
iii. Without loss of generality, we can pick any three of the of the other five vertices to be A, B, and C. So let the 3 vertices farthest to the right of the graph be labelled A, B, C as shown below. Once again, in general the lines OA, OB, and OC could be all red or all green. Let's assume that they are red as shown in the diagram below (if they are green, then the following steps of the proof will be the same but with the colours at each step reversed).



iv. If at least one of AB, BC, or AC is the same colour as OA, OB, and OC (ie. at least one of AB, BC, or AC is red), then a triangle with all 3 sides of the same colour (red) is formed. That is, at least one of $\triangle OAB$, $\triangle OBC$, or $\triangle OAC$ has all 3 sides coloured the same colour (red).



If none of AB, BC, or AC are red, then AB, BC, and AC must all be green! Then, $\triangle ABC$ will have all its sides coloured with the same colour (green). So if none of AB, BC, or AC are the same colour as OA, OB, or OC, then a triangle with all sides coloured the same is still formed.

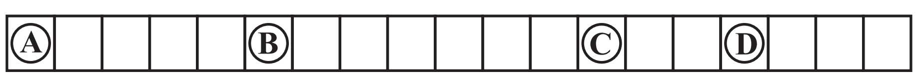


v. Each collection of possible moves made by the two players corresponds to a specific way of colouring the edges of the graph. Since a triangle with all sides coloured the same is formed no matter how we colour in the edges of the graph, this means that one of the two players will win the game no matter what moves are made in the game. Therefore, there are no ties in the two player mathematical game of Sim, and so there must be a winning strategy.

10. ** A game for two players uses four counters on a board which consists of a 20×1 rectangle. The two players alternate turns. A turn consists of moving any one of the four counters any number of squares to the right, but the counter may not land on top of, or move past, any of the other counters. For instance, in the position shown below, the next player could move D one, two or three squares to the right, or move C one or two squares to the right, and so on.

The winner of the game is the player who makes the last legal move. (After this move the counters will occupy the four squares on the extreme right of the board and no further legal moves will be possible.)

In the position shown below, it is your turn. Which move should you make and what should be your strategy in later moves to ensure that you will win the game?



Solution:

The winning strategy in this game is, on your turn, to make the gap between A and B the same number of squares (possibly 0) as the gap between C and D.

Therefore, in the position shown, you should move A two squares to the right, or move D two squares to the right. Your opponent must now move one of the counters so that the two gaps will be different and on your later turns it will always be possible to make the two gaps the same.

As the game continues, D will, sooner or later, be moved to the extreme right square and, on a later move, C will be moved to the last square it can occupy (i.e. the second square from the right). If your opponent moves C to this position, then you will move A to the square immediately to the left of B so that both gaps are now zero. Alternatively, you may move C to its final position yourself if D occupies the last square and your opponent places A on the square adjacent to B. In either case, your opponent is faced with a situation in which B must be moved at least one square. On your turn, you move A the same number of squares to once again reduce the gap to zero. Eventually, your opponent must move B to the square adjacent to C and you then win the game by moving A to its final position.

11. ** **Sprouts**. With a partner, draw one to four dots on a page. Take turns, where each turn consists of drawing a line between two dots (or from a dot to itself) and adding a new dot somewhere along the line. You must follow three rules:

    - The line may be straight or curved, but must not touch or cross itself or any other line
    - The new dot cannot be placed on top of one of the endpoints of the new line. Thus the new spot splits the line into two shorter lines
    - No dot may have more than three lines attached to it. A line from a spot to itself counts as two attached lines and new dots are counted as having two lines already attached to them.

    Whoever makes the last move loses (*Misere* play). Can you determine if there is a winning strategy for 1, 2, 3, and 4 initial dots? [1]

    Solution:

    With perfect play, the first player wins with 1 and 2 initial dots but loses with 3 and 4 initial dots. Proving this is difficult and is not expected of you.

[1] This game is actually played competitively. Notes from `http://en.wikipedia.org/wiki/Sprouts_(game)`