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Grade 7 & 8 Math Circles

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Numbers

Introduction

This week we'll be taking a look through history to see how our knowledge of numbers has evolved over time, in chronological order.

Natural Numbers $\mathbb{N}$

The earliest evidence of counting was found in the year 1960 by a Belgian geographer in the Congo region of Africa. The Ishango bone, a baboon's fibula from 20,000 years ago, was discovered with over 160 markings on it. Because of the systematic nature of the scratches, geologists were certain that the bone was used for counting rather than just being random markings.

Numbers used to count things are called **natural numbers**, and are denoted by $\mathbb{N}$. Natural numbers are positive and can be written without a fractional or decimal component.

$$\mathbb{N} = \{1, 2, 3, 4, 5, ..., 100, ..., 1000, ...\}$$

The need for natural numbers arose as ancient civilizations began increasing in size and record keeping became necessary.

In 4000 BCE the Sumerians of southern Mesopotamia began using tokens to represent numbers. This change made way for arithmetic since you can both add and take away tokens.

1000 years later, Egyptians were the first civilization to develop numerals – different symbols used to represent different numbers. Egyptians invented numerals representing small numbers for slaves and numerals representing larger numbers for aristocrats. Numerals were also critical in the building of pyramids and other structures as they could be used to represent precise measurements.

It was also the Egyptians who first began to use fractions.

Rational Numbers $\mathbb{Q}$

A number that can be represented as the quotient of two natural numbers is called **rational**. That is, a rational number can be expressed as a ratio between p and q, $\frac{p}{q}$, where p and q are natural numbers and q is not zero.

$$\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{N}, q \neq 0\}$$

Fractions can be represented as decimals, so decimals are also rational numbers as long the numbers to the right of the decimal either stop after a few digits, or the same sequence of digits is repeated over and over. Lastly, every natural number is also rational because it can be written as a fraction with a denominator of 1.

Examples

The following are examples of rational numbers.

1. $\frac{1}{2} = 0.5$
2. $\frac{3}{5} = 0.6$
3. $\frac{4}{3} = 1.333...$
4. $\frac{125}{999} = 0.125125125...$
5. $18 = \frac{18}{1} = 18.0$

Because you can convert a fraction like $\frac{12}{27}$ as a repeating decimal, 0.44444444..., you can also convert a repeating decimal into a fraction.

Example

Convert 0.252252252... to a fraction.

Solution

Multiply 0.252252252... by a number that will move the decimal point three points to the right, so that it will reach the end of the first cycle of the pattern.

Let $x = 0.252252252....$

$$1000 \times x = 252.252252...$$

We then subtract 0.252252252... from this number.

$$\begin{array}{rrcl} & 1000 \times x & = & 252.252252252... \\ - & x & = & 0.252252252... \\ \hline & 999 \times x & = & 252 \end{array}$$

Solving this for x:

$$\begin{aligned} 999 \times x &= 252 \\ \frac{999 \times x}{\color{red}{999}} &= \frac{252}{\color{red}{999}} \\ x &= \frac{252}{999} \\ &= \frac{28}{111} \end{aligned}$$

So 0.252252252... can be represented by the fraction $\frac{28}{111}$.

Irrational Numbers $\overline{\mathbb{Q}}$

For over two thousand years the world was certain that all numbers were rational. It wasn't until 500 BCE that Hippasus, a student of the legendary Greek mathematician Pythagoras, discovered that $\sqrt{2}$ could not be written as a fraction and was therefore not a rational number. His announcement was so shocking that he was drowned by Pythagoras's supporters!

It is because of Hippasus that we have **irrational numbers**. Irrational means "not rational", so an irrational number is a number that <u>cannot</u> be written as a ratio between two natural numbers. Irrational numbers also <u>cannot</u> be represented by a decimal where the numbers to the right of the decimal stop or repeat over and over.

Examples

The most famous irrational numbers:

1. $\sqrt{2} = 1.41421356237...$
2. $\pi = 3.14159265359...$
3. $e = 2.71828182845...$
4. $\varphi = 1.61803398875...$

Arabic Numbers

In 500 BCE, while the Egyptians continued to use their numerals, Indians and Romans began developing their own symbols. Roman numerals, which you still see today, used combinations of letters from the Latin alphabet to represent numbers.

Symbol	Value
I	1
V	5
X	10
L	50
C	100
D	500
M	1000

Though Roman numerals were easy to use, they were impractical from a mathematical point of view. Arithmetic was not intuitive when using the numerals, and they could not be used to represent fractions. Instead, Romans simply wrote out the fractions with words if they needed them.

At around the same time, Indians were developing their own system of different symbols for every number from one to nine. Indian numerals were first published on August 28th, 458 AD; the publication also showed that there was familiarity with the decimal system.

These Indian numerals became the Arabic numbers that are used today. Legend says that an Indian ambassador gifted the numerals to the Persian people on a trip to Baghdad. By 1200 AD, Arabic numbers were widely used in North Africa before being brought to Europe by a young Fibonacci.

Zero 0

The number zero did not always exist. As a matter of fact, the first use of zero was found in an Indian temple, dated 876 AD. The concept of zero confused the mathematical world; they asked themselves "How can nothing be something?" The development of zero required not only a symbol that would be able to represent nothing, but the mathematical properties of the number and how to use it in calculations as well.

Because of zero, numbers could be made as small or as large as necessary. As a result of this invention, zero is still considered to be India's greatest contribution to the world.

Integer Numbers $\mathbb{Z}$

Even before the invention of zero, Indians and other cultures understood the concept of negative numbers – they were used to represent debts. It wasn't until the 17th century that mathematicians accepted negative solutions to equations. Before then, negative answers were considered absurd and ignored.

Now that negative numbers are accepted, we can define another number type, as well as redefining rational and irrational numbers.

Positive and negative natural numbers, including zero, are called **integers**, and are denoted by $\mathbb{Z}$.

$$\mathbb{Z} = \{..., -1000, ..., -100, ..., -3, -2, -1, 0, 1, 2, 3, ..., 100, ...1000, ...\}$$

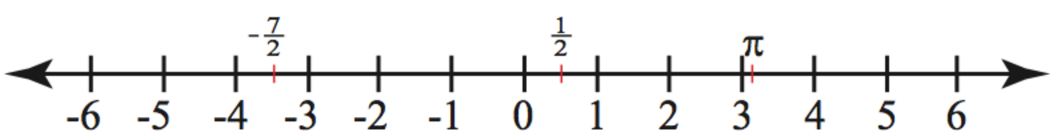
Negative rational numbers are still rational, so now we say that a number that can be represented as the quotient of two integers is called **rational**. That is, a rational number can be expressed as a ratio between p and q, $\frac{p}{q}$, where p and q are integers and q is not zero.

$$\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$$

The definition of irrational numbers can also be extended to include negative numbers: an **irrational number** is a number that cannot be written as a ratio between two *integers*.

Real Numbers $\mathbb{R}$

Every number type we've looked at so far can be classified as **real**. A real number is a value that can be found on a number line.



Descartes was the first to call them real numbers, in the 17th century. He needed to distinguish real numbers from imaginary numbers.

Complex Numbers $\mathbb{C}$

In 1545, as Italian mathematicians were becoming more and more comfortable with negative numbers they found something that they could not explain. They needed to find a number a such that

$$a^2 = -b, \qquad b > 0.$$

But if we remember the rules of multiplying positive and negative numbers, we can see their confusion.

- a **positive** multiplied by a **positive** will result in a **positive** number
- a **negative** multiplied by a **negative** will result in a **positive** number

So any number multiplied by itself should result in a positive number. Then what is the solution to $a^2 = -b, b > 0$?

The **imaginary unit** i had to be defined:

$$i^2 = -1$$

Example

The equation $x^2 = -25$ has solutions $x = 5i$ and $x = -5i$.

CHECK:

$$\begin{aligned}(5i)^2 &= 5i \times 5i \\ &= 5 \times 5 \times i \times i \\ &= 5^2 \times i^2 \\ &= 25 \times (-1) \\ &= -25\end{aligned} \qquad \begin{aligned}(-5i)^2 &= (-5i) \times (-5i) \\ &= (-5) \times (-5) \times i \times i \\ &= (-5)^2 \times i^2 \\ &= 25 \times (-1) \\ &= -25\end{aligned}$$

The imaginary unit allowed mathematicians to extend the real number system into the **complex number** system, denoted by $\mathbb{C}$. A complex number z in standard form is an expression of the form $a + bi$ where a and b are real numbers.

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i^2 = -1\}$$

- $\Re\{a + bi\} = a$ is called the *real part*. When $a = 0$, the number is purely imaginary.
- $\Im\{a + bi\} = b$ is called the *imaginary part*. When $b = 0$, the number is real.

Two imaginary numbers, $z = a + bi$ and $w = c + di$, are *equal* only if $a = c$ and $b = d$. That is, two imaginary numbers are equal if and only if $\Re\{z\} = \Re\{w\}$ and $\Im\{z\} = \Im\{w\}$.

Complex arithmetic was also defined shortly after the imaginary unit was discovered.

Addition:

$$\begin{aligned} z + w &= (a + bi) + (c + di) \\ &= a + bi + c + di \\ &= (a + c) + (b + d)i \end{aligned}$$

Multiplication:

$$\begin{aligned} z \times w &= (a + bi) \times (c + di) \\ &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Division is a little more... complex. It requires us to define the *complex conjugate*:

$$\overline{z} = \overline{a + bi} = a - bi$$

Division:

$$\begin{aligned} \frac{z}{w} &= \frac{z}{w} \times \frac{\overline{w}}{\overline{w}} \\ &= \frac{a + bi}{c + di} \times \frac{\overline{c + di}}{\overline{c + di}} \\ &= \frac{a + bi}{c + di} \times \frac{c - di}{c - di} \\ &= \frac{(a + bi) \times (c - di)}{(c + di) \times (c - di)} \\ &= \frac{ac - adi + bci - bdi^2}{c^2 - cdi + cdi - d^2 i^2} \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} \end{aligned}$$

Examples

1. $$\begin{aligned} (3 + 2i) + (4 - 5i) &= 3 + 2i + 4 - 5i \\ &= 7 - 3i \end{aligned}$$

2. $$\begin{aligned} (3 + 2i) \times (4 - 5i) &= (3)(4) + (3)(-5i) + (2i)(4) + (2i)(-5i) \\ &= 12 - 15i + 8i + 10 \\ &= 22 - 7i \end{aligned}$$

3. $\overline{3 + 2i} = 3 - 2i$

4. $\overline{4 - 5i} = 4 + 5i$

5. $\dfrac{3+2i}{4-5i} = \dfrac{3+2i}{4-5i} \times \dfrac{4+5i}{4+5i}$

$$= \frac{12+15i+8i-10}{16+20i-20i+25}$$

$$= \frac{2+23i}{41}$$

Just as real numbers can be plotted on a number line, complex numbers can be plotted on the *complex plane*. The complex plane has a horizontal real axis and a vertical imaginary axis.

STEPS FOR PLOTTING:
For any complex number $z = a + bi$,

1. Find the real part a on the horizontal real axis.

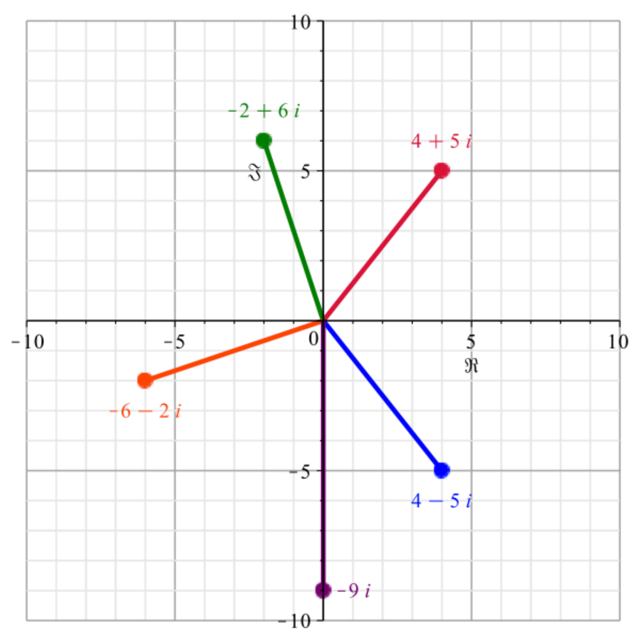
2. Find the imaginary part b on the vertical imaginary axis.

3. Place a point at (a, b) and connect the point to the origin with a straight line.

Examples

Plot the numbers in the plane.

1. $4 + 5i$
2. $4 - 5i$
3. $-2 + 6i$
4. $-6 - 2i$
5. $-9i$

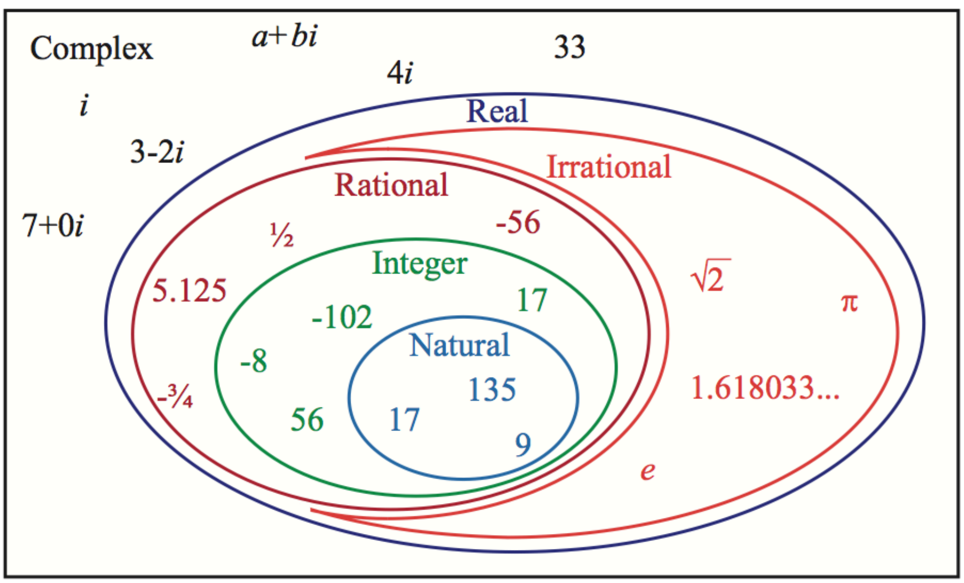


Summary

You have just been introduced to six different types of numbers: natural numbers, rationals, irrationals, integers, real numbers, and complex numbers. It is now time to look for relationships between them all.

- Because integers are positive and negative natural numbers, including zero, every natural number is also an integer.
- Because every integer can be represented by a fraction with a denominator of 1, every integer is also a rational number.
- Every rational number is real, and every real number that isn't rational is irrational.
- Any real number a can be written as $a + 0i$, so every real number is also a complex number.

This is summarized in the diagram below.



Problem Set

1. Classify the following numbers. Check all that apply.

	Natural	Integer	Rational	Irrational	Real	Complex
0	☐	☐	☐	☐	☐	☐
15	☐	☐	☐	☐	☐	☐
$\frac{4}{9}$	☐	☐	☐	☐	☐	☐
$\sqrt{25}$	☐	☐	☐	☐	☐	☐
$\sqrt{3}$	☐	☐	☐	☐	☐	☐
$4-3i$	☐	☐	☐	☐	☐	☐
$5i$	☐	☐	☐	☐	☐	☐
$\sqrt{\frac{1}{4}}$	☐	☐	☐	☐	☐	☐
$-1.3333...$	☐	☐	☐	☐	☐	☐
$3-0i$	☐	☐	☐	☐	☐	☐

2. Is a rational number minus an integer also a rational number?
3. If π is irrational, explain why $\frac{1}{\pi}$ is also irrational.
4. Explain why a rational number multiplied by a rational number results in a rational number.
5. *If I multiplied a rational number by an irrational number, would the product be rational or irrational? Explain.
6. Predict the results of the following expressions. Check all the results that are certain.

	Natural	Integer	Rational	Irrational	Real	Complex
natural×natural	☐	☐	☐	☐	☐	☐
natural×integer	☐	☐	☐	☐	☐	☐
integer×integer	☐	☐	☐	☐	☐	☐
integer×rational	☐	☐	☐	☐	☐	☐
integer×irrational	☐	☐	☐	☐	☐	☐
integer×complex	☐	☐	☐	☐	☐	☐
rational×complex	☐	☐	☐	☐	☐	☐
irrational×complex	☐	☐	☐	☐	☐	☐
integer+irrational	☐	☐	☐	☐	☐	☐

7. Convert the following decimals into fractions (don't need to reduce to lowest terms). For mixed fractions, you don't need to write them as improper fractions.

   (a) $0.\overline{142857}142857...$

   (b) $123.\overline{123}123123123123...$

   (c) $2.\overline{9}9999....$

   (d) * 0.918181818...

   (e) * 0.917181818...

8. Evaluate the following expressions.

   (a) $(2 - 5i) + (1 + i)$

   (b) $(-3 + i) - (-2 - 3i)$

   (c) $3 \times (2 - 5i)$

   (d) $(2 - 5i) \times (1 + i)$

   (e) $(-3 + i) \times (-2 - 3i)$

   (f) $(-3 + i) \times \overline{(-3 + i)}$

   (g) $\overline{1 + i}$

   (h) $\overline{4i - 1}$

   (i) $\dfrac{1 + i}{1 - i}$

   (j) $\dfrac{1}{i}$

   (k) $\dfrac{2 - 5i}{1 + i}$

   (l) $\dfrac{-3 + i}{-2 - 3i}$

9. Evaluate $\overline{\overline{7 - 3i}}$. Generalize the expression for $\overline{\overline{z}}$.

10. *Find a value for k if $\dfrac{2 - ki}{k - i} = 2i$.

11. **The expression $\dfrac{4 - ki}{k - i}$ evaluates to be a real number. Kind a value of k so that this is true.

12. (a) Plot $4 + 3i$ on the complex plane.

    (b) Evaluate $i \times (4 + 3i)$ and plot the result on the same complex plane.

(c) Evaluate $i \times i \times (4+3i)$ and plot the result on the same complex plane.

(d) Evaluate $i \times i \times i \times (4+3i)$ and plot the result on the same complex plane.

(e) Evaluate $i \times i \times i \times i \times (4+3i)$ and plot the result on the same complex plane.

(f) Explain what happens graphically when you multiple a complex number by i.

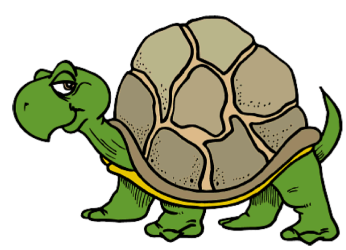
13. *The *modulus* of a complex number $z = a + bi$ is $|z| = |a + bi|$ and represents the *magnitude* of the number. That is, $|z|$ is the length of the line from the origin to the point in the complex plane. Looking at your plots from the previous question, find the formula for $|z|$. (*Hint:* Consider the Pythagorean Theorem.) How is this related to a complex number being multiplied by its conjugate?

14. **Solve the following system of equations if x and y are complex numbers. This requires understanding how to solve systems of equations, unless you are pro enough to solve this just by looking at it (in which case I highly commend you). To learn about solving systems of equations, please refer to last week's lesson.

$$(1+i)x + (1+i)y = 1 + 5i \tag{1}$$

$$(4+2i)x + 4y = 2 - i \tag{2}$$

15. *Harry the mathematical turtle is standing 20 metres from a wall. Because Harry gets tired very easily, it takes him one hour every time to walk halfway toward the wall. This means that it takes him an hour to walk 10 metres, then an hour to walk 5 metres, and so on. Will he ever reach the wall? And if he does, how long will it take him?



Answers

1. See solutions.

2. Yes. See solutions.

3. See solutions.

4. See solutions.

5. Irrational. See solutions.

6. See solutions.

7. (a) $\frac{1}{7}$ (b) $123\frac{123}{999}$ (c) 3 (d) $\frac{101}{110}$ (e) $\frac{10809}{11000}$

8. (a) $3 - 4i$ (b) $-1 + 4i$ (c) $6 - 15i$
   (d) $7 - 3i$ (e) $9 + 7i$ (f) 10
   (g) $1 - i$ (h) $-4i - 1$ (i) i
   (j) $-i$ (k) $\frac{-3 - 7i}{2}$ (l) $\frac{3 - 11i}{13}$

9. $7 - 3i$; $\overline{\overline{z}} = z$

10. $k = 0$

11. $k = -2$ or $k = 2$

12. (a) See solutions for all plots.
    (b) $-3 + 4i$
    (c) $-4 - 3i$
    (d) $3 - 4i$
    (e) $4 + 3i$
    (f) Counter-clockwise rotation by 90° about the origin.

13. $|z| = \sqrt{a^2 + b^2} = \sqrt{z \times \overline{z}}$

14. $x = -\frac{9}{2} + 5i$ and $y = \frac{15}{2} - 3i$

15. Theoretically, Harry will never reach the wall.