UNIVERSITY OF WATERLOO
Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Grade 7 & 8 Math Circles

October 15/16, 2013

Numbers - Solutions

Problem Set

1.

	Natural	Integer	Rational	Irrational	Real	Complex
0	☐	☒	☒	☐	☒	☒
15	☒	☒	☒	☐	☒	☒
$\frac{4}{9}$	☐	☐	☒	☐	☒	☒
$\sqrt{25}$	☐	☒	☒	☐	☒	☒
$\sqrt{3}$	☐	☐	☐	☒	☒	☒
$4-3i$	☐	☐	☐	☐	☐	☒
$5i$	☐	☐	☐	☐	☐	☒
$\sqrt{\frac{1}{4}}$	☐	☐	☒	☐	☒	☒
$-1.3333...$	☐	☐	☒	☐	☒	☒
$3-0i$	☒	☒	☒	☐	☒	☒

2. Yes it is. Let $\frac{a}{b}$ be a rational number, and let n be any integer. Then

$$\begin{aligned}\frac{a}{b} - n &= \frac{a}{b} - \frac{bn}{b} \\ &= \frac{a-bn}{b}\end{aligned}$$

But since $\frac{a}{b}$ is a rational number, a and b are integers. So $bn = b \times n$ is an integer, and therefore $a - bn$ is also an integer. This means that $\frac{a-bn}{b}$ is a ratio of two integers – it is a rational number.

3. We use Sherlock Holmes' reasoning. There are two possible cases:

   (a) $\frac{1}{\pi}$ is irrational

   (b) $\frac{1}{\pi}$ is rational

   We want to show that (b) is false. To do this, we assume it is true, and show that this leads to an impossibility.

   Assume (b) is true, i.e. $\frac{1}{\pi}$ is a rational number. Then $\frac{1}{\pi} = \frac{m}{n}$, where m and n are integer with $n \neq 0$ (this is just the definition of a rational number).

   Isolate for π.

$$\begin{aligned} \frac{1}{\pi} &= \frac{m}{n} \\ \frac{1}{\pi}\times\pi &= \frac{m}{n}\times\pi \\ 1 &= \frac{m\pi}{n} \\ 1\times n &= \frac{m\pi}{n}\times n \\ n &= m\pi \\ \frac{n}{m} &= \frac{m\pi}{m} \\ \frac{n}{m} &= \pi \end{aligned}$$

   So if we assume that $\frac{1}{\pi}$ **is rational**, then π itself is a rational number. But this is impossible, since we know π **is irrational**. Therefore $\frac{1}{\pi}$ cannot be rational.

   The only other possibility, (however improbable), is that $\frac{1}{\pi}$ **is irrational**. This provides a convincing argument.

4. Let a, b, c, and d all be integers, and b and d not be equal to zero. Thus $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers since they are the ratio between two integers. Their product is

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

   But ac and bd are products of integers, so they themselves are integers. Hence $\frac{ac}{bd}$ is a rational number.

5. The product is irrational. To see this, we apply the same reasoning used in the previous question. Let us assume that the product is **rational** at first. If this is true, let $r = \frac{p}{q}$ be the rational number, and s be the irrational number.

   We are claiming that rs is a rational number. So $rs = \frac{m}{n}$. But since $r = \frac{p}{q}$, rs also equals $\left(\frac{p}{q}\right)(s) = \frac{ps}{q}$.

   Since rs equals two different things, those two different things must be equal, i.e.

$$\frac{ps}{q} = \frac{m}{n}$$

   If we isolate for s, we get

$$\begin{aligned} \frac{ps}{q} &= \frac{m}{n} \\ \frac{ps}{q}\times q &= \frac{m}{n}\times q \\ ps &= \frac{qm}{n} \\ \frac{ps}{p} &= \frac{qm}{np} \\ s &= \frac{qm}{np} \end{aligned}$$

   This shows that s is actually a ratio of two integers, qm and np. But this is impossible, since s is an **irrational** number. This must mean that the original assumption that rs was rational is not possible. The only remaining possibility is that rs is in fact, **irrational**.

6.

	Natural	Integer	Rational	Irrational	Real	Complex
natural×natural	☒	☒	☒	☐	☒	☒
natural×integer	☐	☒	☒	☐	☒	☒
integer×integer	☐	☒	☒	☐	☒	☒
integer×rational	☐	☐	☒	☐	☒	☒
integer×irrational	☐	☐	☐	☒	☒	☒
integer×complex	☐	☐	☐	☐	☐	☒
rational×complex	☐	☐	☐	☐	☐	☒
irrational×complex	☐	☐	☐	☐	☐	☒
integer+irrational	☐	☐	☐	☒	☒	☒

7. (a) $0.\overline{142857}142857...$

The digits 142857 are being repeated.

Let $x = 0.\overline{142857}142857....$ The pattern is six digits long, so multiply by $10^6 = 1\ 000\ 000$.

Then

$$\begin{array}{rrcl} & (1\ 000\ 000)x & = & 142857.142857142857... \\ - & x & = & 0.142857142857... \\ \hline & 999\ 999x & = & 142857 \\ & \frac{999\ 999x}{999\ 999} & = & \frac{142857}{999\ 999} \\ & x & = & \frac{1}{7} \end{array}$$

(b) $123.\overline{123}123123123123...$

Note $123.\overline{123}123123123123... = 123 + 0.\overline{123}123123123123...$ so we only need to deal with the decimal tail.

Let $x = 0.123123...$ The pattern is three digits long, so multiply by $10^3 = 1\ 000$.

Then

$$\begin{array}{rrcl} & (1\ 000)x & = & 123.123123... \\ - & x & = & 0.123123... \\ \hline & 999x & = & 123 \\ & x & = & \dfrac{123}{999} \end{array}$$

So $0.123123... = \dfrac{123}{999}$. Hence $123.123123... = 123 + 0.123123... = 123\ \dfrac{123}{999}$.

(c) $2.\overline{9}9999....$

In a manner similar to (b), we see that $2.999... = 3$. (This is not a mistake, this is correct).

(d) $0.918181818...$

Let $x = 0.918181818...$

Our strategy is to multiply x by powers of 10, to get two different numbers that have the same decimal tail.

For the first number, note that the pattern is two digits long, but we also want to skip over the first digit 9 (we want the decimal tail to match up) so multiply by $10^3 = 1\ 000$. This gives $(1\ 000)x = 918.181818....$

For the second number, we cannot simply subtract x from $(1\ 000)x$, as this does not get rid of the decimal tail completely. The reason is that we have that pesky 9 digit. To solve this, multiply x by 10, to get $10x = 9.181818....$

So $(1\ 000)x$ and $10x$ have the same decimal tail. We can now subtract them.

Then

$$\begin{array}{rrcl} & (1\ 000)x & = & 918.181818... \\ - & 10x & = & 9.181818... \\ \hline & 990x & = & 909 \\ & x & = & \dfrac{909}{990} \end{array}$$

In reduced form (optional) this is $\dfrac{101}{110}$.

(e) 0.917181818...

Let $x = 0.917181818...$

Our strategy is to multiply x by powers of 10, to get two different numbers that have the same decimal tail.

For the first number, note that the pattern is two digits long, but we also want to skip over the first three digit 917 (we want the decimal tail to match up) so multiply by $10^5 = 100\ 000$. This gives $(100\ 000)x = 91718.1818....$

For the second number, we cannot simply subtract x from $(100\ 000)x$, as this does not get rid of the decimal tail completely. The reason is that we have those pesky 917 digits. To solve this, multiply x by 1000, to get $1000x = 917.1818....$

So $(100\ 000)x$ and $1000x$ have the same decimal tail. We can now subtract them.

Then

$$\begin{array}{rrcl} & (100\ 000)x & = & 91718.1818... \\ - & 1000x & = & 917.1818... \\ \hline & 99000x & = & 90801 \\ & x & = & \dfrac{90801}{99000} \end{array}$$

In reduced form (optional) this is $\dfrac{10809}{11000}$.

8. (a) $(2 - 5i) + (1 + i) = 2 - 5i + 1 + i$

$= 3 - 4i$

(b) $(-3+i)-(-2-3i) = -3+i+2+3i$
$$= -1+4i$$

(c) $3\times(2-5i) = 3\times 2 - 3\times 5i$
$$= 6-15i$$

(d) $(2-5i)\times(1+i) = (2)(1)+(2)(i)+(-5i)(1)+(-5i)(i)$
$$= 2+2i-5i+5$$
$$= 7-3i$$

(e) $(-3+i)\times(-2-3i) = (-3)(-2)+(-3)(-3i)+(i)(-2)+(i)(-3i)$
$$= 6+9i-2i+3$$
$$= 9+7i$$

(f) $(-3+i)\times\overline{(-3+i)} = (-3+i)\times(-3-i)$
$$= (-3)(-3)+(-3)(-i)+(i)(-3)+(i)(-i)$$
$$= 9+3i-3i+1$$
$$= 10$$

(g) $\overline{1+i} = 1-i$

(h) $\overline{4i-1} = -4i-1$

(i) $$\frac{1+i}{1-i} = \frac{1+i}{1-i}\times\frac{1+i}{1+i}$$
$$= \frac{(1)(1)+(1)(i)+(i)(1)+(i)(i)}{(1)(1)+(1)(i)+(-i)(1)+(-i)(i)}$$
$$= \frac{1+i+i-1}{1+i-i+1}$$
$$= \frac{2i}{2}$$
$$= i$$

(j) $$\frac{1}{i} = \frac{1}{i}\times\frac{-i}{-i}$$
$$= \frac{-i}{1}$$
$$= -i$$

(k) $$\frac{2-5i}{1+i} = \frac{2-5i}{1+i} \times \frac{1-i}{1-i}$$
$$= \frac{(2)(1)+(2)(-i)+(-5i)(1)+(-5i)(-i)}{(1)(1)+(1)(-i)+(i)(1)+(i)(-i)}$$
$$= \frac{2-2i-5i-5}{1-i+i+1}$$
$$= \frac{-3-7i}{2}$$
$$= -\frac{3}{2} - \frac{7}{2}i$$

(l) $$\frac{-3+i}{-2-3i} = \frac{-3+i}{-2-3i} \times \frac{-2+3i}{-2+3i}$$
$$= \frac{(-3)(-2)+(-3)(3i)+(i)(-2)+(i)(3i)}{(-2)(-2)+(-2)(3i)+(-3i)(-2)+(-3i)(3i)}$$
$$= \frac{6-9i-2i-3}{4-6i+6i+9}$$
$$= \frac{3-11i}{13}$$
$$= \frac{3}{13} - \frac{11}{13}i$$

9. $\overline{\overline{7-3i}}$ is the conjugate of the conjugate of $7-3i$. To evaluate this, evaluate $\overline{7-3i}$ and then find the conjugate of the result.

$$\overline{7-3i} = 7+3i$$

So

$$\overline{\overline{7-3i}} = \overline{7+3i}$$
$$= 7-3i$$

Thus, the conjugate of the conjugate of a complex number is equal to the original complex number:

$$\overline{\overline{z}} = z$$

10. Multiply both sides by $k - i$

$$\frac{2-ki}{k-i}\times(k-i) = 2i\times(k-i)$$
$$2-ki = 2i(k-i)$$
$$2-ki = (2k)i - 2(i^2)$$
$$2-ki-2 = (2k)i + 2-2$$
$$\frac{-ki}{i} = \frac{2ki}{i}$$
$$-k = 2k$$
$$-k+k = 2k+k$$
$$0 = 3k$$
$$\frac{0}{3} = \frac{3k}{3}$$
$$0 = k$$

Hence $k = 0$ satisfies $\frac{2-ki}{k-i} = 2i$.

11. Let a be a real number. We need $\frac{4-ki}{k-i} = a$. Multiply both sides by $k - i$ to get

$$\frac{4-ki}{k-i}\times(k-i) = a\times(k-i)$$
$$4-ki = a(k-i)$$
$$4-ki = ak - ai$$
$$4-ki+ai = ak-ai+ai$$
$$4-ki+ai = ak$$
$$4-ki+ai-ak = ak-ak$$
$$4-ki+ai-ak = 0$$
$$(4-ak)+(a-k)i = 0$$

The left hand side of the equation is a complex number, but it must equal 0. So we need $(4 - ak) = 0$ and $(a - k) = 0$.

This is a system of equations with two unknowns!

$$4 - ak = 0 \tag{1}$$
$$a - k = 0 \tag{2}$$

Use (2) to solve for k. $a - k = 0$, so $k = a$.

Substitute this into equation (1) to get $4 - ak = 4 - (k)(k) = 4 - k^2 = 0$. To solve for k,

$$\begin{aligned} 4 - k^2 &= 0 \\ 4 - k^2 + k^2 &= 0 + k^2 \\ 4 &= k^2 \end{aligned}$$

So $k = -2$ or $k = 2$.

12. (a) Entire plane is below.

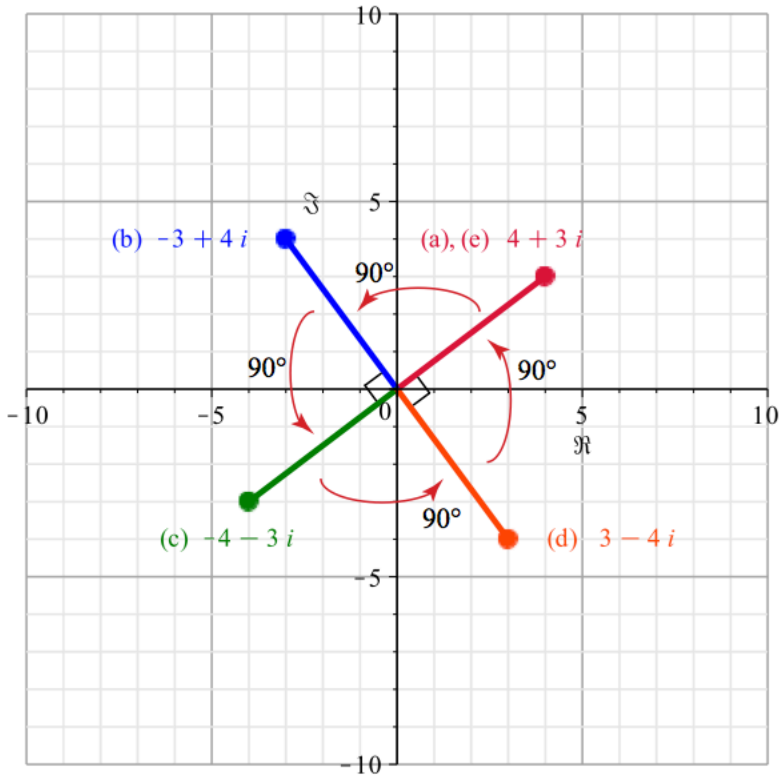
    (b) $i \times (4 + 3i) = i \times 4 + i \times 3i$
    $= -3 + 4i$

    (c) $i \times i \times (4 + 3i) = i \times (-3 + 4i)$
    $= i \times (-3) + i \times 4i$
    $= -4 - 3i$

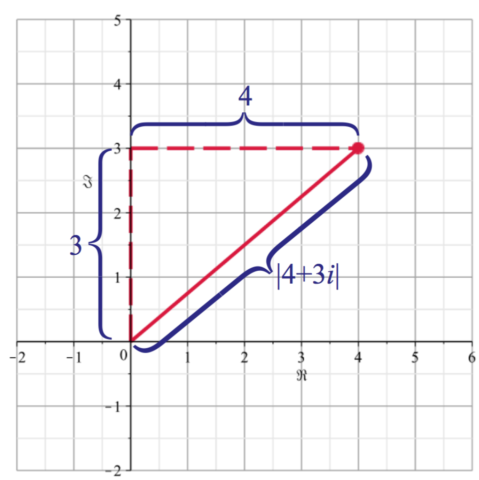
    (d) $i \times i \times i \times (4 + 3i) = i \times (-4 - 3i)$
    $= i \times (-4) + i \times (-3i)$
    $= 3 - 4i$

    (e) $i \times i \times i \times i \times (4 + 3i) = i \times (3 - 4i)$
    $= i \times 3 + i \times (-4i)$
    $= 4 + 3i$

    (f) As you can see in the image below, multiplying a complex number by i results in a counter-clockwise rotation by $90°$ about the origin.



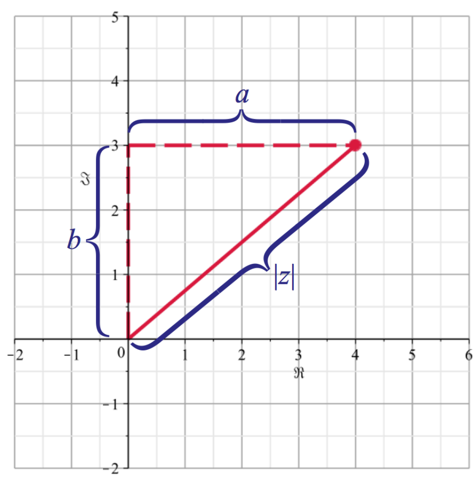
13. We'll investigate this by looking at the portion of the plane containing $4+3i$. To apply the Pythagorean Theorem, we must work with a right triangle, as shown below.



The Pythagorean Theorem states $a^2 + b^2 = c^2$, where a and b are the lengths of the two shortest sides of the triangle, and c is the length of the longest side of the triangle (the diagonal side). In our case, $a = 4$, $b = 3$, and $c = |4+3i|$. Plugging these numbers into the formula:

$$
\begin{aligned}
4^2 + 3^2 &= |4+3i|^2 \\
16 + 9 &= |4+3i|^2 \\
\sqrt{25} &= \sqrt{|4+3i|^2} \\
5 &= |4+3i|
\end{aligned}
$$

Now we must generalize this to find $|z|$.



$$
\begin{aligned}
a^2 + b^2 &= |z|^2 \\
\sqrt{a^2+b^2} &= \sqrt{|z|^2} \\
\sqrt{a^2+b^2} &= |z|
\end{aligned}
$$

Thus the modulus of a complex number $z = a + bi$ is defined by $|z| = \sqrt{a^2+b^2}$.

For the second part of the question, evaluate $z \times \overline{z}$.

$$
\begin{aligned}
z \times \overline{z} &= (a+bi) \times \overline{(a+bi)} \\
&= (a+bi) \times (a-bi) \\
&= (a)(a) + (a)(-bi) + (bi)(a) + (bi)(-bi) \\
&= a^2 - abi + abi + b^2 \\
&= a^2 + b^2
\end{aligned}
$$

Thus $|z| = \sqrt{a^2+b^2} = \sqrt{z \times \overline{z}}$.

14. The system:

$$(1+i)x + (1+i)y = 1+5i \tag{1}$$
$$(4+2i)x + 4y = 2-i \tag{2}$$

Use (1) to isolate for y.

$$\begin{aligned}(1+i)x + (1+i)y &= 1+5i\\ (1+i)y &= 1+5i-(1+i)x\\ y &= \frac{1+5i}{1+i} - x\\ y &= 3+2i-x\end{aligned}$$

Substitute into equation (2) and solve for x.

$$\begin{aligned}(4+2i)x + 4(3+2i-x) &= 2-i\\ 4x+(2i)x+12+8i-4x &= 2-i\\ (2i)x &= -10-9i\\ x &= -\frac{10}{2i} - \frac{9}{2}\\ &= -\frac{9}{2}+5i\end{aligned}$$

Substitute the expression above into $y = 3+2i-x$ to solve for x.

$$\begin{aligned}y &= 3+2i-(-\frac{9}{2}+5i)\\ &= \frac{15}{2}-3i\end{aligned}$$

Hence $x = -\frac{9}{2}+5i$ and $y = \frac{15}{2}-3i$.

15. Theoretically, Harry will never reach the wall. This is because no matter how close he is to the wall, he will still have to travel halfway. Look at the table on the next page.

Hour	Distance Travelled	Total Distance Travelled	Distance Remaining
0	0	0	20
1	10	10	10
2	5	15	5
3	2.5	17.5	2.5
4	1.25	18.75	1.25
$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	0.01953125	19.98046875	0.01953125
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The "Distance Remaining" column will never be zero because dividing any non-zero number by two can never equal zero. That is, the equation $\frac{x}{2} = 0$ is only true when $x = 0$.

Note: This shows us that there are infinite rational numbers between any two integers.