Problem Solving, Part 3

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Outline

- Review of exercises from last week
- Intuition
- Write down everything you know
- A "computer science" "problem"
- Work on problems

Review of exercises from last week

Question 1:

Prove that $n^2 + 3n + 5$ is never divisible by 121 for any positive integer n.

Suppose $n^2 + 3n + 5$ is divisible by 121. ← contradiction!

$\Rightarrow \quad n^2 + 3n + 5 = 121 \cdot k$ for some k (integer)

$\Rightarrow \quad n^2 + 3n + (5 - 121k) = 0$

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9 - 4(1)(5 - 121k)}}{2}$$

$$= \frac{-3 \pm \sqrt{9 - 20 + 484k}}{2} = \frac{-3 \pm \sqrt{11(44k - 1)}}{2}$$

$\therefore \quad 44k - 1 = 11t$ for some t integer but $44k = 11(4)$ and $11 \nmid -1$

Review of exercises from last week

Question 2:
Prove that $n! + 1$ and $(n+1)! + 1$ are always relatively prime numbers, for any positive integer n.

Suppose $p \mid n!+1$ and $p \mid (n+1)!+1$ (p is prime)

$\Rightarrow p \mid \left((n+1)! \cancel{+1} - (n! \cancel{+1})\right)$

$\Rightarrow p \mid \left((n+1)! - n!\right)$

$\Rightarrow p \mid \left(n!\left((n+1) - 1\right)\right) \Rightarrow p \mid n(n!)$

$p \mid n$

$\Rightarrow p \mid n!$

contradiction

Review of exercises from last week

Question 3:
Prove that if $ax^2 + bx + c = 0$ has real solutions for x and that a, b, c are not zero, then a, b and c cannot form a geometric sequence.

Suppose a, b, c are a geometric sequence

$b = ar \quad c = ar^2$ for some r

Discriminant $\Rightarrow b^2 - 4ac \geq 0$

$(ar)^2 - 4(a)(ar^2) \geq 0$

$-3a^2r^2 \geq 0$

< 0 $\quad \geq 0$ $\quad \geq 0$

contradiction

Review of exercises from last week

Question 4: Euclid 2011 Q5(a)

What is the largest two-digit number that becomes 75% greater when its digits are reversed?

$n = 10a + b$ $\qquad$ $N = 10b + a$

$\frac{7}{4} n = N$

$7n = 4N$

$7(10a+b) = 4(10b+a)$

$66a = 33b$

$2a = b$

digits $a = 1..4$

$b = 8$

Review of exercises from last week

Question 5: Euclid 2010 8(a)

Oi-Lam tosses three fair coins and removes all of the coins that come up heads. George then tosses the coins that remain, if any. Determine the probability that George tosses exactly one head.

Cases:

Oi-Lam	3 heads	$\Rightarrow$	no chance	
Oi-Lam	2 heads	$\Rightarrow$	1 coin	$\Rightarrow \frac{1}{2}$
	1 head	$\Rightarrow$	2 coins	HH HT TH TT $= \frac{1}{2}$
	0 heads	$\Rightarrow$	3 coins	HHH HHT HTH THH TTH THT HTT TTT $\frac{3}{8}$

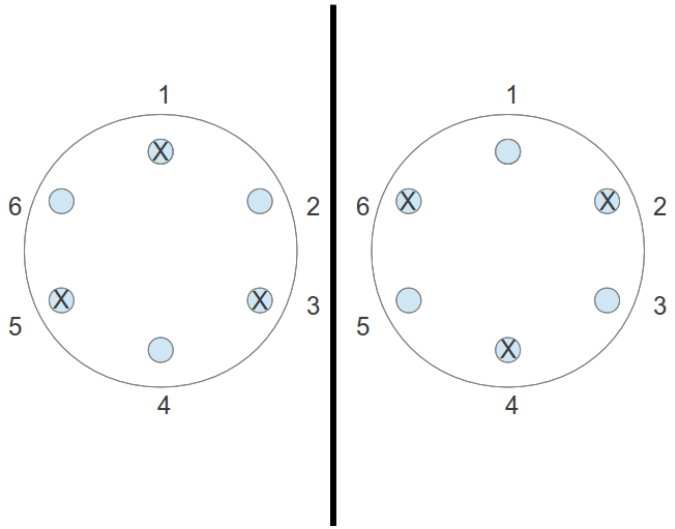
prob of exactly one head George

$$\frac{1}{8}\cdot 0 + \frac{3}{8}\cdot\frac{1}{2} + \frac{3}{8}\cdot\frac{1}{2} + \frac{1}{8}\cdot\frac{3}{8} = \frac{27}{64}$$

Review of exercises from last week

Question 6: Euclid 2009 10(a)+(b)

Suppose there are n plates equally spaced around a circular table. Ross wishes to place an identical gift on each of k plates, so that no two neighbouring plates have gifts. Let $f(n, k)$ represent the number of ways in which he can place the gifts. For example $f(6, 3) = 2$, as shown below:



Review of exercises from last week

Question 6(a)
Determine the value of $f(7, 3)$.

Review of exercises from last week

Question 6(b)

Prove that $f(n,k) = f(n-1,k) + f(n-2,k-1)$ for all integers $n \geq 3$ and $k \geq 2$.

Intuition

- Your gut instinct is often wrong

Intuition

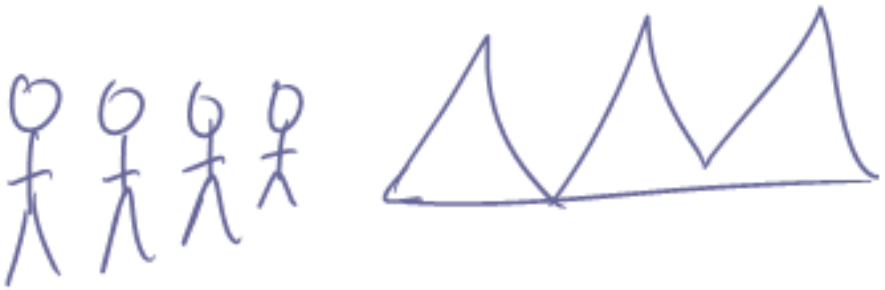
- Your gut instinct is often wrong
- Use your brain instead

Intuition

Four math students have to cross a bridge at night to escape from their maniacal teacher. The bridge is damaged, and so can take at most two students at a time. When two students cross together, they have to travel at the speed of the slowest. The four students have only one flashlight between them: to cross safely, the flashlight has to go across each time. If the four students individually take 1, 2, 4, and 6 minutes to cross the bridge, what is the shortest time needed for all four to cross safely?

1. 1,2
2. 2
3. 4,6
4. 1
5. 1,2

13



1. 1,2	2
2. 1	1
3. 1,4	4
4. 1	1
5. 1,6	6

Write down what you know

- Often on problems, you get stuck.

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Write down what you know

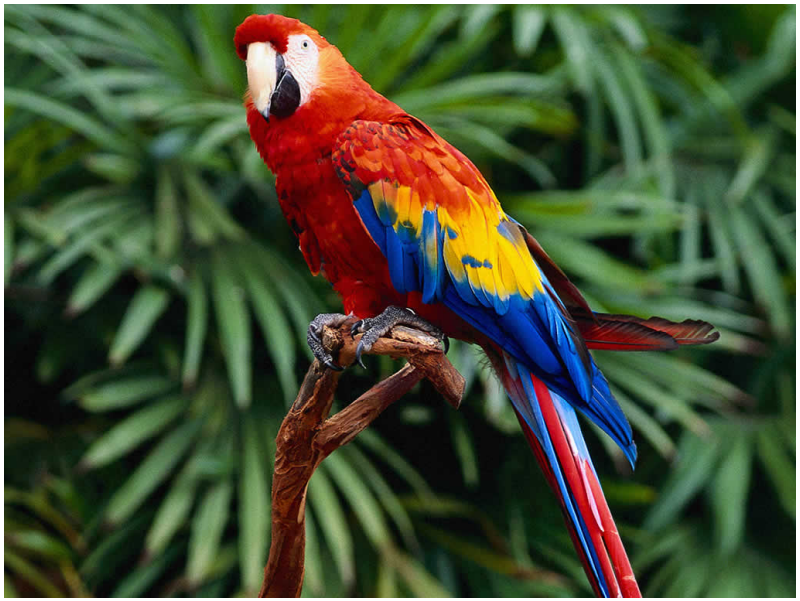
- Often on problems, you get stuck.
- One way to get unstuck is to write down all the facts you know at a certain point in your reasoning.
- Try different approaches and combinations of the information that you have already.

A neat computer science problem

The 23rd International Olympiad in Informatics

THAILAND

2011



2011 IOI, Day 2, Problem 3: Parrots

Problem:
Send a message composed of N integers, each of which is between 0 and 255.

Difficulty:
The sequence order may be scrambled.

Task:
Send a sequence of integers so that the original message can be recovered.

Demo:

`http://www.ioi2011.or.th/hsc/demo/parrots/index.html`

Subtask A

- $1 \leq N \leq 16$ ← at most 16 integers to send
- Each encoded integer sent must be between 0 and 65535.

$2^6 = 64$

$2^{10} = 1024$

$2^{16} = 65536$

Idea:

Subtask A

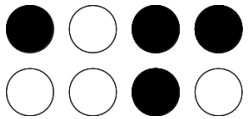
- $1 \leq N \leq 16$
- Each encoded integer sent must be between 0 and 65535.

Idea:

- find a way to include positions with the integers of the original message (call the positions 0, 1, 2, ...).
- For example, to send the message 13, 223, 7, somehow send $(13, 0)$, $(223, 1)$, $(7, 2)$.

A bit about bits

- All data is a sequence of bits
- There are many meanings to a sequence
- For example: 10110010 could mean
  - $178 = 128 + 32 + 16 + 2$
  - 2,3,0,2
  - 2
  - a small black and white image:



Constraints tell us something

178 = 10110010

255 = 11111111

- We know we have only N numbers to send

$$1 \leq \overset{\text{values}}{\cancel{N}} \leq 255 = 2^8 - 1$$

- We send only the numbers from 0 to $65535 = 2^{16} - 1$
- $$\frac{16}{2} = 8$$

8 bits 8 bits

Solution to Subtask A

1 2 3 4

256

00000001 00000000

Using 16 bits, we use:

- 8 bits to send the number
- 8 bits to send the position (though we only really need 4 of these bits to represent one of the 16 positions in the sequence)

0 1 2 3 15

01017 0255

Subtask B

- $1 \leq N \leq 16$
- Each encoded integer is between 0 and **255**

8 bit numbers

17 37 4 203 18

0117

Subtask B

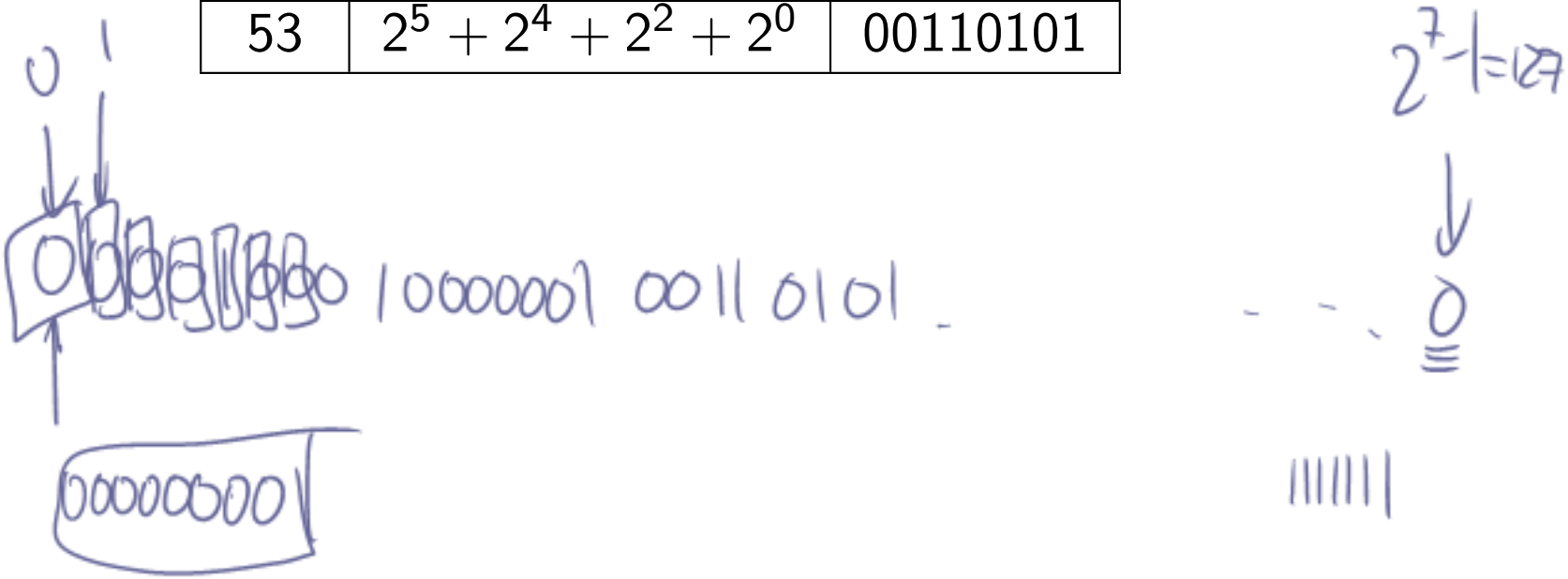
- $1 \leq N \leq 16$
- Each encoded integer is between 0 and **255**
- Previous solution no longer works
  - only have 8 bits per sent integer
  - need to capture the order with every encoded integer

Solution to Subtask B

Solution to Subtask B

View the sequence only as bits

8	2^3	00001000
129	$2^7 + 2^0$	10000001
53	$2^5 + 2^4 + 2^2 + 2^0$	00110101



Solution to Subtask B

- Since there are at most 16 numbers, and each number needs only 8 bits, there is a sequence of at most $128 = 2^7$ bits.

$16 \cdot 8 = 128$

Solution to Subtask B

- Since there are at most 16 numbers, and each number needs only 8 bits, there is a sequence of at most $128 = 2^7$ bits.
- Send each bit itself with its position
  - 1 bit for the "bit"
  - 7 bits for the position

Subtask C

- $1 \leq N \leq \mathbf{32}$
- each encoded integer is between 0 and 255

Subtask C

index

- $1 \leq N \leq \mathbf{32}$
- each encoded integer is between 0 and 255
- The previous solution no longer works

$32 \cdot 8 = 256 = 2^8$

so need all 8 bits for the position

255

0123

010110 - - - - - - - - - - 0

Solution to Subtask C

- We have 32 numbers, each of which can be represented with 8 bits.
- In total, we have a sequence of $256 = 2^8$ bits

Solution to Subtask C

- We have 32 numbers, each of which can be represented with 8 bits.
- In total, we have a sequence of $256 = 2^8$ bits
- Only send the positions of bits equal to 1
- all 8 bits can be used for the positions of the 1
- any unsent positions must be 0

Subtask D

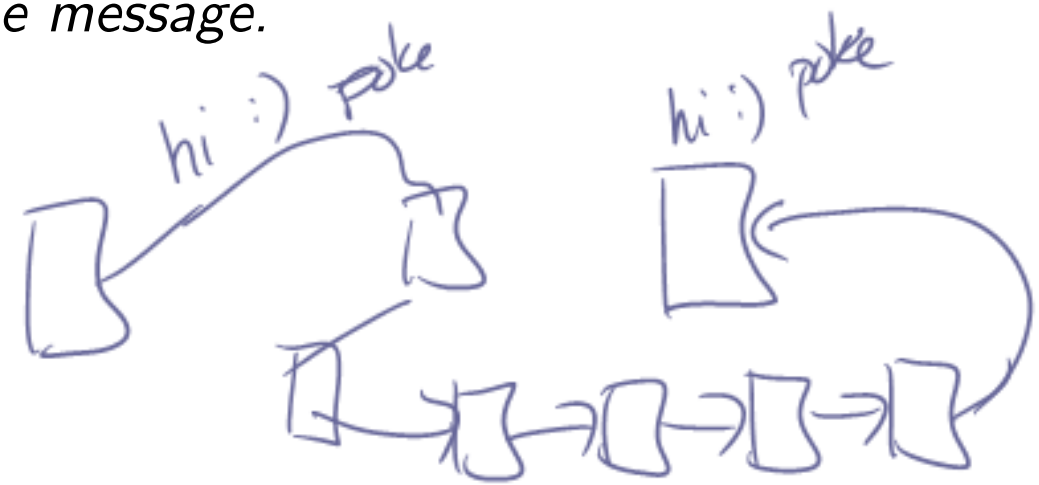
- $1 \leq N \leq \mathbf{64}$
- Each encoded integer is between 0 and 255
- Make the encodings as small as possible

Subtask D

- $1 \leq N \leq \mathbf{64}$
- Each encoded integer is between 0 and 255
- Make the encodings as small as possible
- The previous solution no longer works
- *There is an encoding that is never more than about 4.08 times larger than the message.*

packet routing

ipv6
ipv4



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- Solve the problem being asked
- Don't lean on intuition
- Look for symmetry and patterns
- Choose good notation
- Practice, practice, practice