Math Circles: Diophantine Equations II (Geometric interpretation of linear Diophantine equations)

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Geometric interpretation

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1. $6x + 4y = 5$

2. $6x + 4y = 2$

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2. $6x + 4y = 2$

   Since $\gcd(6, 4) = 2$, and 2 *does divide* 2, this equation does have an integer solution. We can find one by inspection: $x = 1$, $y = -1$. In fact, there are infinitely many integer solutions.

Let's examine these two situations geometrically...

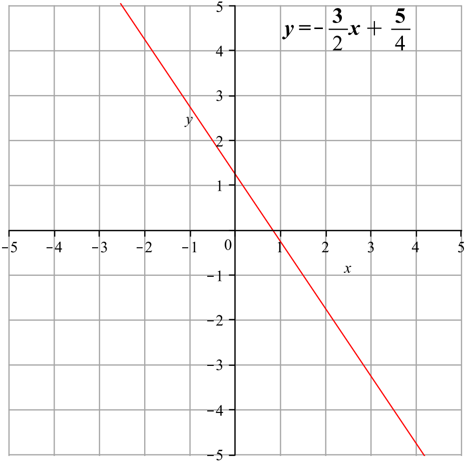
The equation $6x + 4y = 5$

Notice that this is the equation of a line. We can rearrange the equation to get:

$$\begin{aligned} 6x + 4y &= 5 \\ 4y &= -6x + 5 \\ y &= -\frac{6}{4}x + \frac{5}{4} \\ y &= -\frac{3}{2}x + \frac{5}{4} \end{aligned}$$

Since this is a line, there are infinitely many pairs of numbers x and y that satisfy $y = -\frac{3}{2}x + \frac{5}{4}$, but we know that there are no *integer* pairs, because the equation $6x + 4y = 5$ has no integer solutions.

On a graph...

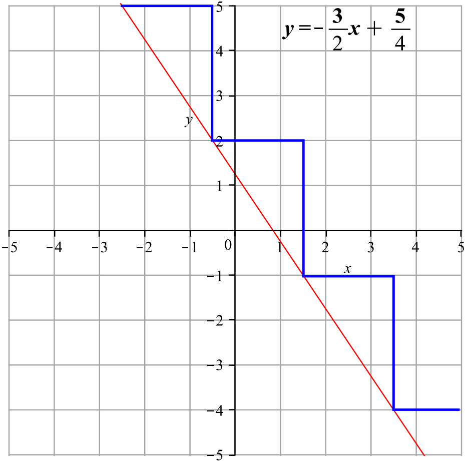


Lattice points are points in the plane with integer x and y coordinates.

These will correspond to integer solutions of our linear Diophantine equation.

The line misses all the lattice points shown in this graph.

The equation $6x + 4y = 5$



The line misses *all* lattice points.

The equation has no integer solutions.

The equation $6x + 4y = 2$

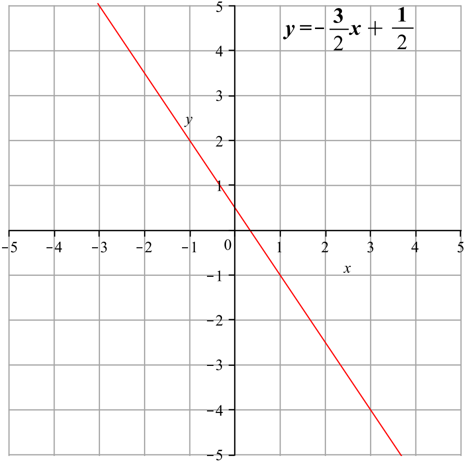
Again, this is the equation of a line. We can rearrange the equation to get:

$$\begin{aligned} 6x + 4y &= 2 \\ 4y &= -6x + 2 \\ y &= -\frac{6}{4}x + \frac{2}{4} \\ y &= -\frac{3}{2}x + \frac{1}{2} \end{aligned}$$

This second line has the same slope as the first line, but with a different y intercept.

Since this is a line, there are infinitely many pairs of numbers x and y that satisfy $y = -\frac{3}{2}x + \frac{1}{2}$. Let's convince ourselves with a picture that there are also infinitely many integer solutions.

The equation $6x + 4y = 2$



The line hits many lattice points on this graph.

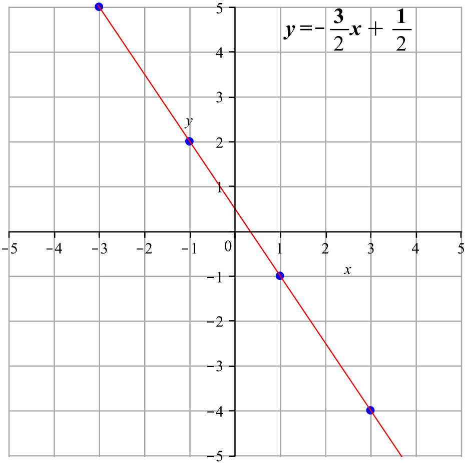
$(-3, 5)$

$(-1, 2)$

$(1, -1)$

$(3, -4)$

The equation $6x + 4y = 2$



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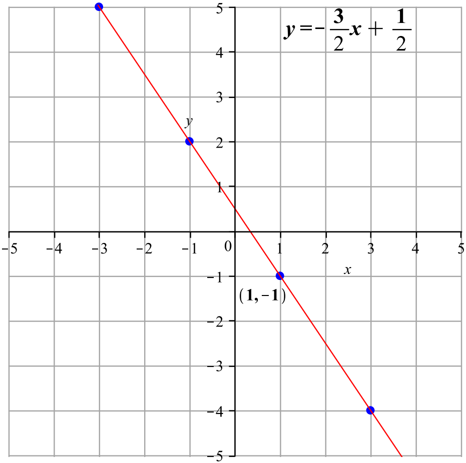
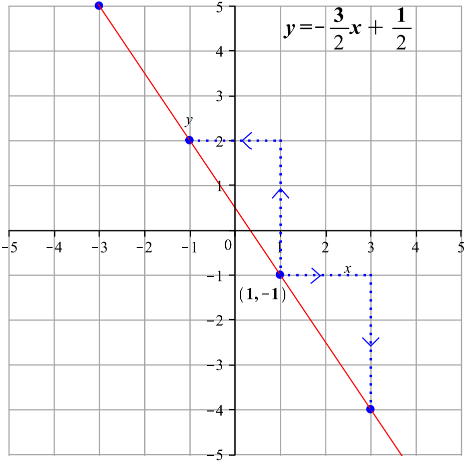


If we pick one lattice point, then we can find its "neighbouring lattice points" as follows.

The equation $6x + 4y = 2$



If we pick one lattice point, then we can find its "neighbouring lattice points" as follows.

$$(1-2, -1+3) = (-1, 2)$$

$$\uparrow$$

$$(1, -1)$$

$$\downarrow$$

$$(1+2, -1-3) = (3, -4)$$

General method

Given one solution x_0,y_0 to an equation $ax + by = c$, how do we find its "neighbouring solutions"?

Assuming that $a, b \neq 0$, let's rearrange this equation as follows:

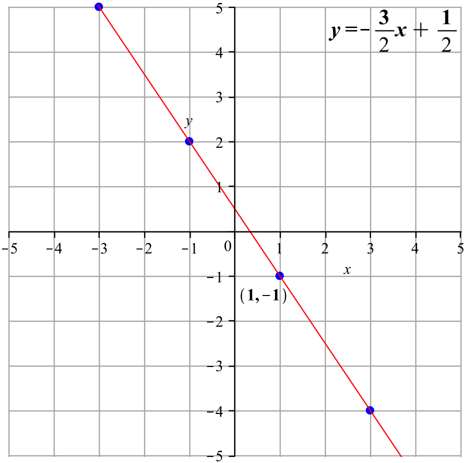
$$\begin{aligned} ax + by &= c \\ by &= -ax + c \\ y &= -\frac{a}{b}x + \frac{c}{b} \end{aligned}$$

giving us a slope of $m = -\frac{a}{b}$.

So we might be tempted so say "move a units up and b units to the left" or "move b units to the right and a units down".

This will indeed find us another solution, but which one?

Back to our example

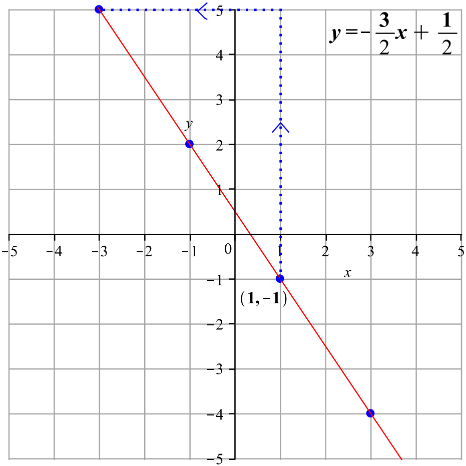


Let's return to the example $6x + 4y = 2$. Then we have

$$y = -\frac{6}{4}x + \frac{2}{4} = -\frac{3}{2}x + \frac{1}{2}$$

with $a = 6$ and $b = 4$.

Back to our example



Let's return to the example $6x + 4y = 2$. Then we have

$$y = -\frac{6}{4}x + \frac{2}{4} = -\frac{3}{2}x + \frac{1}{2}$$

with $a = 6$ and $b = 4$.

If we simply move $a = 6$ units up and $b = 4$ units to the left then we may miss lattice points!

Neighbouring solutions

$$ax + by = c \quad \rightsquigarrow \quad y = -\frac{a}{b}x + \frac{c}{b}$$

While $-\frac{a}{b}$ is the slope of the line, in order to use this fraction to find ALL lattice points, we need to make sure that we consider the fraction $\frac{a}{b}$ in lowest terms.

$$\text{The fraction } \tfrac{a}{b} \text{ in lowest terms} = \frac{\left(\dfrac{a}{\gcd(a,b)}\right)}{\left(\dfrac{b}{\gcd(a,b)}\right)}$$

To find the "neighbouring solutions":

- Move right $\dfrac{b}{\gcd(a,b)}$ units, and down $\dfrac{a}{\gcd(a,b)}$ units, or
- Move up $\dfrac{a}{\gcd(a,b)}$ units, and left $\dfrac{b}{\gcd(a,b)}$ units.

Example

Given that $x_0 = 1$, $y_0 = -1$ is one solution to the equation $6x + 4y = 2$, find the two "neighbouring" integer solutions.

Solution: Using the method from the previous slide:

First we find $\frac{a}{\gcd(a,b)} = \frac{6}{2} = 3$ and $\frac{b}{\gcd(a,b)} = \frac{4}{2} = 2$.

To find the "neighbouring" solutions we do the following:

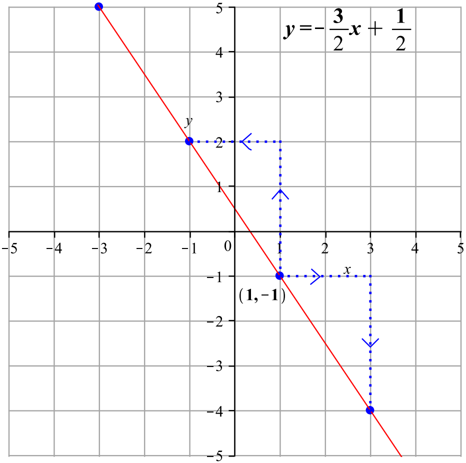
- $x = x_0 + \frac{b}{\gcd(a,b)} = 1 + 2 = 3$ *"right 2 units"*

  $y = y_0 - \frac{a}{\gcd(a,b)} = -1 - 3 = -4$ *"down 3 units"*

- $x = x_0 - \frac{b}{\gcd(a,b)} = 1 - 2 = -1$ *"left 2 units"*

  $y = y_0 + \frac{a}{\gcd(a,b)} = -1 + 3 = 2$ *"up 3 units"*

Confirm graphically



Given one solution $x_0 = 1$, $y_0 = -1$, the "next lattice points" are:

$$(1-2, -1+3) = (-1, 2)$$

$$\uparrow$$

$$(1, -1)$$

$$\downarrow$$

$$(1+2, -1-3) = (3, -4)$$

Example

Given that $x_0 = 9$, $y_0 = -20$ is one integer solution to the equation $483x + 217y = 7$, find the two "neighbouring" integer solutions.

Solution: We have $a = 483$ and $b = 217$ and $\gcd(483, 217) = 7$ from earlier calculations.

$$\frac{a}{\gcd(a,b)} = \frac{483}{7} = 69 \text{ and } \frac{b}{\gcd(a,b)} = \frac{217}{7} = 31.$$

To find the "neighbouring" solutions we do the following:

- $x = x_0 + \dfrac{b}{\gcd(a,b)} = 9 + 31 = 40$ *"right 31 units"*

  $y = y_0 - \dfrac{a}{\gcd(a,b)} = -20 - 69 = -89$ *"down 69 units"*

- $x = x_0 - \dfrac{b}{\gcd(a,b)} = 9 - 31 = -22$ *"left 31 units"*

  $y = y_0 + \dfrac{a}{\gcd(a,b)} = -20 + 69 = 49$ *"up 69 units"*

Check! $483(40) + 217(-89) = 7$ and $483(-22) + 217(49) = 7$.

Finding all solutions

Suppose that x_0, y_0 is one integer solution to the linear Diophantine equation $ax + by = c$, and let $d = \gcd(a, b)$. Then the **full set** of integer solutions for the equation is given by

$$\boxed{x = x_0 + n\left(\frac{b}{d}\right), \quad y = y_0 - n\left(\frac{a}{d}\right) \quad \text{where } n \text{ is any integer.}}$$

Note that $n = 1$ corresponds to a "neighbour" of x_0, y_0:

$$\begin{aligned} x &= x_0 + 1\left(\frac{b}{d}\right) \\ &= x_0 + \frac{b}{d} \end{aligned} \qquad \begin{aligned} y &= y_0 - 1\left(\frac{a}{d}\right) \\ &= y_0 - \frac{a}{d} \end{aligned}$$

Also $n = -1$ corresponds to the other "neighbour" of x_0, y_0:

$$\begin{aligned} x &= x_0 + (-1)\left(\frac{b}{d}\right) \\ &= x_0 - \frac{b}{d} \end{aligned} \qquad \begin{aligned} y &= y_0 - (-1)\left(\frac{a}{d}\right) \\ &= y_0 + \frac{a}{d} \end{aligned}$$

Example (6)

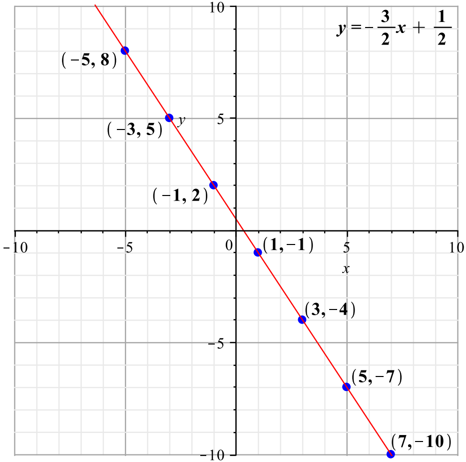
Find ALL integer solutions to the equation $6x + 4y = 2$.

Solution:
We know that $a = 6$, $b = 4$ and $\gcd(a,b) = \gcd(6,4) = 2 = d$.
Since $x_0 = 1$, $y_0 = -1$ is one solution to the given equation, the full set of integer solutions is given by

$$\begin{aligned} x &= x_0 + n\left(\tfrac{b}{d}\right) & \qquad y &= y_0 - n\left(\tfrac{a}{d}\right) \\ &= 1 + n\left(\tfrac{4}{2}\right) & &= -1 - n\left(\tfrac{6}{2}\right) \\ &= 1 + 2n & &= -1 - 3n \end{aligned}$$

1. Every integer n produces a particular solution $x = 1 + 2n$, $y = -1 - 3n$ to the equation, and
2. Every integer solution to the equation is of the form $x = 1 + 2n$, $y = -1 - 3n$ for some integer n.

General solution $x = 1 + 2n$, $y = -1 - 3n$



$n = -3$

$n = -2$

$n = -1$

$n = 0$ (original solution)

$n = 1$

$n = 2$

$n = 3$