Intermediate Math Circles
February 5, 2014
Contest Preparation I

Welcome back. It has been a long break and now we are coming up on the Pascal, Cayley and Fermat Contests. The contests are all written on the same day: Thursday, February 20, 2014. The deadline for registration is February 6. If you are not registered, approach the supervisor in your school or the math department head too see if there is something you can do....TOMORROW!

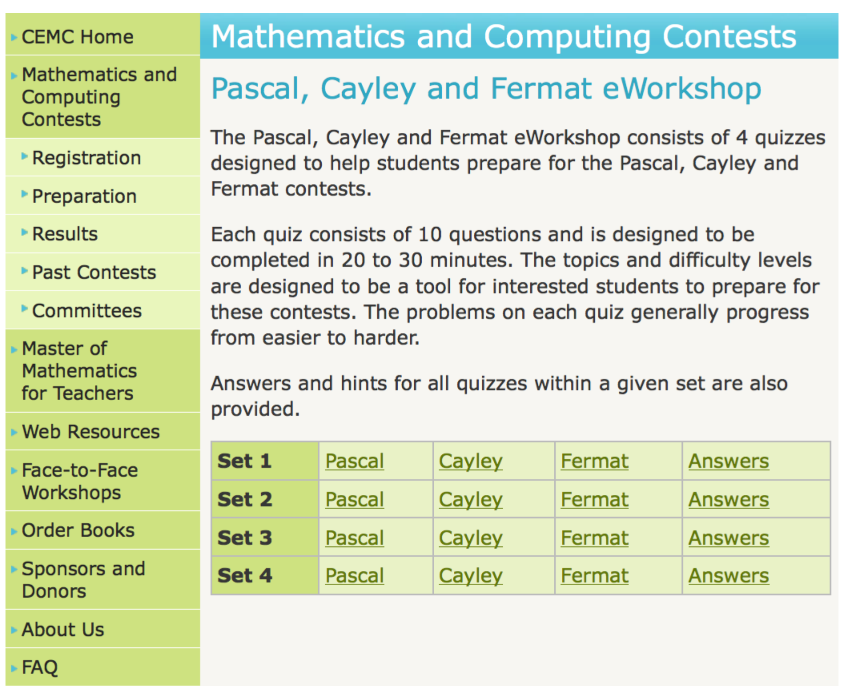
Each contest has 25 multiple choice problems. The problems are meant to gradually increase in difficulty, so problem 1 should be the easiest and problem 25 the hardest.

The problems are split up into three sections. Section A has ten questions each worth 5 points, section B has ten questions each worth 6 points and section C has five questions each worth 8 points. In addition, any question left unanswered is worth 2 points up to a maximum of 20 points. This is important to consider as it means guessing is not always rewarded.

Most of the problems that we will look at have been taken from past Pascal and Cayley contests.

The key to success on any math contest is do lots and lots of problems. Writing a Math contest is different than solving math problems in general. The contest is timed! By doing many old contests you will pick up some tricks and techniques which will help you to write a good contest.

Go to http://www.cemc.uwaterloo.ca/contests/pcf_eWorkshop.html for the Pascal, Cayley, Fermat E-workshop.



You can also find old contests at: http://www.cemc.uwaterloo.ca/contests/past_contests.html#pcf

North and South America PDF

Pascal (Grade 9)

back to top

Contests	2011 2010 2009 2008 2007 2006 2005 2004 2003 2002 2001 2000 1999 1998 1997 Download all as single PDF
Solutions	2011 2010 2009 2008 2007 2006 2005 2004 2003 2002 2001 2000 1999 1998 1997
Results	2011 2010 2009 2008 2007 2006 2005 2004 2003 2002 2001

Cayley (Grade 10)

Contests	2011 2010 2009 2008 2007 2006 2005 2004 2003 2002 2001 2000 1999 1998 1997 Download all as single PDF
Solutions	2011 2010 2009 2008 2007 2006 2005 2004 2003 2002 2001 2000 1999 1998 1997
Results	2011 2010 2009 2008 2007 2006 2005 2004 2003 2002 2001

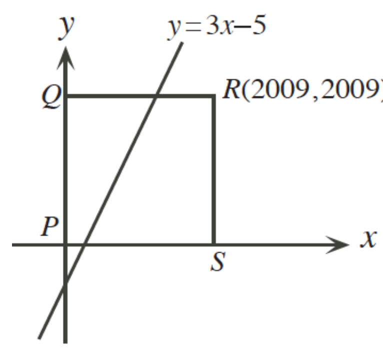
Fermat (Grade 11)

Contests	2011 2010 2009 2008 2007 2006 2005 2004 2003 2002 2001 2000 1999 1998 1997 Download all as single PDF
Solutions	2011 2010 2009 2008 2007 2006 2005 2004 2003 2002 2001 2000 1999 1998 1997

WARM-UP #1

A *lattice point* is a point with integer coordinates. (For example, $(1, 4)$ is a lattice point but $(\frac{3}{2}, 4)$ is not.) The line $y = 3x - 5$ passes through the square $PQRS$ as shown in the diagram. If the coordinates of R are $(2009, 2009)$, then the number of lattice points on the line which are inside the square is

(A) 666 (B) 667 (C) 668 (D) 669 (E) 670



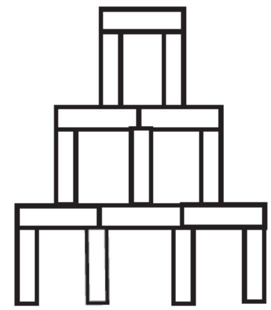
Solution: $0 \leq y \leq 2009$ since the point must be inside the square. Plugging $y = 0$ into $y = 3x - 5$ we get $x = \frac{5}{3}$. x is an integer $\geq \frac{5}{3}$ so the smallest value for $x = 2$. Plugging $y = 2009$ into $y = 3x - 5$ we get $x = \frac{2014}{3} \doteq 671.3$. x is an integer $\leq \frac{2014}{3}$ so the smallest value for $x = 671$. So for every x-value from 2 to 671 there is a lattice point on the line $y = 3x - 5$ inside the square. From 1 to 671 there are 671 numbers so from 2 to 671 there are 670 numbers and the correct answer is (E).

WARM-UP #2

From Practice Pascal #1 Question 9 in the PCF EWorkshop

Dean is building a tower with blocks as shown in the diagram. The tower shown has three stories and uses 15 blocks. How many blocks are required for a tower of 80 stories?

(A) 400 (B) 6399 (C) 6496 (D) 6560 (E) 6723



Solution #1

Count the horizontal and vertical blocks in the 80 rows.
There are $1 + 2 + 3 + \cdots + 79 + 80$ blocks placed flat in the first 80 stories.
There are $2 + 3 + 4 + \cdots + 80 + 81$ blocks standing vertically in the first 80 stories.

We can use the formula for the sum of the first n integers $\frac{n(n+1)}{2}$
So $1 + 2 + 3 + \cdots + 79 + 80 = \frac{80(81)}{2} = 3240$.
And $2+3+4+\cdots+80+81 = 1+2+3+4+\cdots+80+81-1 = \frac{81(82)}{2} - 1 = 3321 - 1 = 3320$.

The total number of blocks in 80 stories is $3240 + 3320 = 6560$ and the answer is (D).

Solution #2

This solution looks at patterns. In the first story there are 3 blocks. In the first two stories there are 8 blocks. In the first three stories there are 15 blocks.

Story Number	Number of Blocks	Observation
1	3	$4 - 1 = 2^2 - 1$
2	8	$9 - 1 = 3^2 - 1$
3	15	$16 - 1 = 4^2 - 1$

It would appear that the number of blocks is related to the story number such that the number of blocks equals (Story $\#+1)^2 - 1$. So in a tower with 80 stories there would be $81^2 - 1 = 6561 - 1 = 6560$. The answer is (D).

To start, work on **Problem Set 1**.

Answers to Problem Set 1:

1 D	2 A	3 D	4 B	5 D	6 D
7 E	8 D	9 E	10 B	11 E	12 D

Read Carefully

Were there any of the questions from Problem Set 1 that you had to read especially carefully?

In #2 you were asked for the average **side length** not average area.
In #8 you were asked for the total number of students not just the number of girls or boys.
In #11 you had to carefully keep track of what was given.
In #12 you were given the sum of the three side lengths not just the hypotenuse.

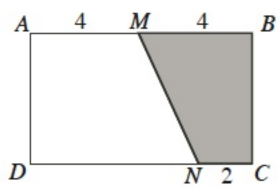
Narrowing Down the Choices

Were there any of the questions from Problem Set 1 that you could have easily eliminated choices from?

In #2 the longest side is $\sqrt{169} = 13$ so the average cannot be 24, 39, or 32, so rule out C,D,E.
In #4 you probably can quickly rule out an average above 85 since the highest number of students got 80 and the range of the data is quite narrow.

Two new ones to try:

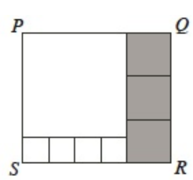
1. In the diagram, the area of rectangle $ABCD$ is 40. The area of $MBCN$ is

   (A) 15 (B) 10 (C) 30 (D) 12 (E) 16

   We can see in the diagram that M is the midpoint of AB. We can also see that NC is one quarter of the length of DC. As a result, we can tell that the area of the shaded region is more than one quarter, but less than one half of the area of the rectangle. This means the area of the shaded region is more than 10 but less than 20. Right away we can discount (B) 10 and (C) 30 as answers. The answer is: A (2007 Cayley #9).

2. Rectangle $PQRS$ is divided into eight squares, as shown. The side length of each shaded square is 10. What is the length of the side of the largest squares?

   (A) 18 (B) 24 (C) 16 (D) 23 (E) 25

   From the diagram, we can see that the side of the large square is longer than two side lengths of the shaded squares. This means the side length of the large square must be more than $2 \times 10 = 20$. We can discount (A) 18 and (C) 16 as answers. The answer is: B (2009 Cayley #16).

Plug in the Given Values!

1. When $x = 9$, which of the following has the largest value?
   (A) $\sqrt{x}$ (B) $\frac{x}{2}$ (C) $x - 5$ (D) $\frac{40}{x}$ (E) $\frac{x^2}{20}$

   Instead of considering each of the answers with the variable x still in it, simply plug in $x = 9$ and calculate the value of each answer. The answer is B (2006 Pascal #11)

2. If $a = 7$ and $b = 13$, the number of even positive integers less than ab is
   (A) $\frac{ab-1}{2}$ (B) $\frac{ab}{2}$ (C) $ab - 1$ (D) $\frac{a+b}{4}$ (E) $(a-1)(b-1)$

   Once again, substitute $a = 7$ and $b = 13$ into the answers and calculate the value of each answer. The answer is A (2008 Cayley #9). Note here that you could also elimanate choices.

Try writing out a few cases!

1. When three consecutive positive integers are multiplied together, the answer is always
   (A) odd (B) a multiple of 6 (C) a multiple of 12 (D) a multiple of 4
   (E) a multiple of 5

   Three possible consecutive positive integers are 1, 2, and 3. $1 \times 2 \times 3 = 6$. 6 is not odd, 6 is not a multiple of 12 or a multiple of 4 or a multiple of 5. The only possibility left is a multiple of 6 so the answer is (B) (This was from 2009 Cayley #8)

2. The increasing list of five different integers $\{3,4,5,8,9\}$ has a sum of 33. How many increasing lists of five different single-digit positive integers have a sum of 33?
   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

   Try using the three largest positive single-digit integers, 7, 8, and 9. The sum is currently at 24 leaving 9 more to go to get to 33. There are two possibilities: 3, 6 and 4, 5. Try using the two largest positive single-digit integers and 6 but not 7. Using 6,8,9 the sum is 23, leaving 10 to go. There is no way to sum to 10 using two of the possible digits. Now try 6,7,9 but not 8. These digits sum to 22 leaving 11 to go. In very short order, it can be shown that there are only two possible cases and so the answer is (B).

To start, work on **Problem Set 2 and 3**.