Intermediate Math Circles
February 5, 2014
Contest Preparation I

Answers to Problem Sets 1 and 2

Answers to Problem Set 1:

1. D	2. A	3. D	4. B
5. D	6. D	7. E	8. D
9. E	10. B	11. E	12. D

Answers to Problem Set 2:

1. B	2. E	3. B	4. E
5. D	6. E	7. D	8. E
9. D	10. D	11. A	12. C
13. C	14. E	15. A	

** Important: The solutions in this document are solutions to the problems whose solutions cannot be found online at the CEMC website. All problems which have labels of its origin (i.e. which contest it is taken from) have their solutions online in the corresponding contest section on the CEMC website.

Solutions:

Problem Set 1:

6) In order for $7k52$ to be divisible by 12, it must be divisible by 2 and 3 since the prime factorization of 12 is $12 = 2 \times 2 \times 3$. Since the last digit of $7k52$ is even, it is divisible by 2. In order for $7k52$ to be divisible by 3, the sum of the digits must be divisible by 3. The sum of the digits is $14 + k$. The values of k can be between 0 to 9.
If $k = 1$ The sum of the digits is 15 which is divisible by 3.
If $k = 4$ The sum of the digits is 18 which is also divisible by 3.
If $k = 7$ The sum of the digits is 21 which is also divisible by 3.
So there are 3 different values of k for which $7k52$ is divisible by 12 and the answer is D.

Problem Set 2:

1) The next number after 2722 that has three digits with the same number is 2777. So the least number of kilometers that must be travelled to reach a number with three digits of the same number is 55 which is between 50 and 100. Thus, the answer is B.

3) To find the value of $0 * 0$, substitute 0 into the definition of $a * b$ for a and b.
$a * b = (a + 1)(b - 1)$
$0 * 0 = (0 + 1)(0 - 1)$
$= (1) \times (-1)$
= -1
So the answer is B.

5) We see that 7 is a divisor of 777. Every time we subtract 7, the resulting number will still be divisible by 7. (If $a \div b = q$, then $(a - b) \div b = (a \div b) - (b \div b) = q - 1$, hence $a - b$ also divides b). Out of the numbers in the list, only 42 divides 7. Hence the answer is D.

6) The total cost of the dinner is $\$8.43 + \$13.37 + \$2.46 = \$24.26$.
If Pat and Chirs split the dinner evenly, the cost of one share of the dinner is $\$24.26 \div 2 = \$12.13$. But since Pat has already paid \$2.46, he owes Chris

7) Let x represent the value of the smallest integer. Then since the integers are consecutive, we can represent the other four integers by $(x+1)$, $(x+2)$, $(x+3)$, and $(x+4)$.
Then the sum of the integers is:
$x + (x+1) + (x+2) + (x+3) + (x+4) = 75$
$5x + 10 = 75$
$5x = 75 - 10$
$5x = 65$
$x = 65 \div 5$
$x = 13$
Thus, the value of the smallest integer is 13 and the value of the largest integer is $13 + 4 = 17$.
So the sum of the smallest and largest integers is $13 + 17 = 30$. Therefore, the answer is D.

8) Let x_1 represent the number of points Megan scored in the first game.
Let x_2 represent the number of points Megan scored in the second game.
Let x_3 represent the number of points Megan scored in the third game.
Let x_4 represent the number of points Megan scored in the fourth game.
Since after the first three games Megan scored an average of 18 points per game,
$\frac{x_1+x_2+x_3}{3} = 18$
$x_1 + x_2 + x_3 = 18 \times 3$
$x_1 + x_2 + x_3 = 54$
Then, after the fourth game her points per game average drops to 17. So,
$\frac{x_1+x_2+x_3+x_4}{4} = 17$
$x_1 + x_2 + x_3 + x_4 = 17 \times 4$
$x_1 + x_2 + x_3 + x_4 = 68$
Since we know that $x_1 + x_2 + x_3 = 54$, we can substitute this into the equation.
$54 + x_4 = 68$
$x_4 = 68 - 54$
$x_4 = 14$
Therefore, Megan scored 14 points in the fourth game and the answer is E.

10) The smallest perfect square over 10 is $4^2 = 16$; the largest perfect square under 200 is $14^2 = 196$. For any odd number between 4 and 14, its square is also odd. (odd $\times$ odd $=$ odd); so its square+1 is even, which are not primes. This leaves us

divisible by 5; $12^2 = 144$, by the same argument, 145 is not a prime. We only have $4, 6, 10, 14$ left. We check and get that $17, 37, 101, 197$ are all primes. These are the only such n. Hence the answer is D.

Aside: (In general, it is a difficult problem to determine whether a very large number is prime or not (by very large, I'm talking a number with millions of digits). But for small numbers, a feasible and trivial method to determine whether it's prime is to try to divide it by **every prime** less than its **square root**. If none of them divides this number, then your number is prime. Can you explain why this method is valid?)