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Grade 6/7/8 Math Circles

April 1/2, 2014

Modular Arithmetic - Solutions

Exercises I

1. Fill in the blanks:

   (a) $55 = 6 \ (mod \ \ 7)$

   (b) $2048 = 2 \ (mod \ \ 3)$

   (c) $406 = 406 \ (mod \ \ 1056)$

2. What congruence classes exist for modulo 3?

   [0],[1],[2]

   (a) List 3 numbers that belong to each of these classes.

   Answer may vary.

3. What congruence classes exist for modulo 7?

   [0],[1],[2],[3],[4],[5],[6]

   (a) List 3 numbers that belong to each of these classes.

   Answer may vary.

Exercises II

1. $X \equiv 6 \ (mod \ \ 7)$ and $Y \equiv 16 \ (mod \ \ 7)$.

   (a) What is X + Y equivalent to in modulo 7?

$$\begin{aligned} X \equiv 6 \ (mod \ \ 7) + Y \equiv 16 \ (mod \ \ 7) &= (X+Y) \equiv (6+16) \ (mod \ \ 7) \\ &= (X+Y) \equiv 22 \ (mod \ \ 7) \\ &= (X+Y) \equiv 1 \ (mod \ \ 7) \end{aligned}$$

(b) What is X - Y equivalent to in modulo 7?

$$\begin{aligned} X \equiv 6\ (mod\ \ 7) - Y \equiv 16\ (mod\ \ 7) &= (X - Y) \equiv (6 - 16)\ (mod\ \ 7) \\ &= (X - Y) \equiv -10\ (mod\ \ 7) \\ &= (X - Y) \equiv 4\ (mod\ \ 7) \end{aligned}$$

(c) What is Y - X equivalent to in modulo 7?

$$\begin{aligned} Y \equiv 16\ (mod\ \ 7) - X \equiv 6\ (mod\ \ 7) &= (Y - X) \equiv (16 - 6)\ (mod\ \ 7) \\ &= (Y - X) \equiv 10\ (mod\ \ 7) \\ &= (Y - X) \equiv 3\ (mod\ \ 7) \end{aligned}$$

(d) What is X × Y equivalent to in modulo 7?

$$\begin{aligned} X \equiv 6\ (mod\ \ 7) \times Y \equiv 16\ (mod\ \ 7) &= (X \times Y) \equiv (6 \times 16)\ (mod\ \ 7) \\ &= (X \times Y) \equiv 96\ (mod\ \ 7) \\ &= (X \times Y) \equiv 5\ (mod\ \ 7) \end{aligned}$$

Problem Set

NOTE: A leap year occurs every four years. A leap year occurs on any year that is divisible by 4 (**ex:** 4, 8, 12 ... 1996, 2000, 2004, 2008, 2012)

1. Solve the following:

   (a) What is 84 (mod 9)?

   3 (mod 9)

   (b) What is 52 (mod 5)?

   2 (mod 5)

   (c) What is -4 (mod 10)?

   6 (mod 10)

2. Create the following tables:

   (a) Addition table for modulo 7

+	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[0]	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[1]	[1]	[2]	[3]	[4]	[5]	[6]	[0]
[2]	[2]	[3]	[4]	[5]	[6]	[0]	[1]
[3]	[3]	[4]	[5]	[6]	[0]	[1]	[2]
[4]	[4]	[5]	[6]	[0]	[1]	[2]	[3]
[5]	[5]	[6]	[0]	[1]	[2]	[3]	[4]
[6]	[6]	[0]	[1]	[2]	[3]	[4]	[5]

   (b) Multiplication table for modulo 7

+	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[0]	[0]	[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[2]	[0]	[2]	[4]	[6]	[1]	[3]	[5]
[3]	[0]	[3]	[6]	[2]	[5]	[1]	[4]
[4]	[0]	[4]	[1]	[5]	[2]	[6]	[3]
[5]	[0]	[5]	[3]	[1]	[6]	[4]	[2]
[6]	[0]	[6]	[5]	[4]	[3]	[2]	[1]

3. I celebrated my 21st birthday on Wednesday, July 27th, 2011. On what day of the week was I born? (Don't forget about leap years!)

   $21 \times 365 = 7665$
   Leap years occurred in 2008, 2004, 2000, 1996, and 1992, so we add 5 more days to get total of 7670 days.
   $7670 \equiv 5 \pmod 7$, and remember since we are looking into to past we are going backwards 5 days from Wednesday, therefore I was born on a Friday.

4. One year on Venus lasts 225 Earth days. Alysha is 13 years and 83 days old. How many days until her next Venusian birthday? How old will she be turning (in Venusian years)? Omit leap years for simplicity.

   Alysha is $13 \times 365 + 83 = 4828$ earth days old.
   225 goes into 4828 21 times with 103 left over, so Alysha is 21 years and 103 days old on Venus. She will celebrate her 22^{nd} Venusian birthday in $225 - 103 = 122$ days.

5. It is 8:00 AM in our 24 hour world. What time is it in a 3 hour world?

   $8 \equiv 2 \pmod 3$, therefore it is 2:00.

6. Using a standard 52 card deck I deal all the cards out to Vince, Tim, and myself. Were the cards dealt evenly?

   No, $53 \equiv 1 \pmod 3$ so somebody has one more card than the other two people.

7. Luc is facing West, he rotates 1260° clockwise. What direction is he now facing? (**Note:** A circle has 360 degrees)

   We are working with the modulus 360 because there are 360° in one rotation.
   $1260 \equiv 180 \pmod{360}$ so Luc is now facing 180° clockwise from West, which is East.

8. ** 1 year on Jupiter is equal to approximately 12 Earth years. On what day of the week did you celebrate your 1^{st} Jovian (or Jupiterian) birthday? (If you haven't turned 1 on Jupiter yet, calculate on which day of the week your 1^{st} birthday will fall)

   Answers will vary.

9. *** Tim counted the loonies in her pocket. When she put them in groups of 4, she had 2 loonies left over. When she put them in groups of 5, she had one loonie left over. If Tim has more than 10 loonies, what is the smallest possible number of loonies she could have?

   We are looking for the smallest possible x greater than 10 such that
   $x \equiv 2 \pmod 4$ and $x \equiv 1 \pmod 5$. Putting both congruence classes side by side we get:

$\equiv 2 \pmod 4$	$\equiv 1 \pmod 5$
10	11
14	16
18	21
22	**26**
26	
30	
34	

Therefore the smallest possible number of loonies Tim could have is 26.