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Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles
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Counting I - Solutions

Exercises I

1. Barry the Bookworm has 5 biology books, 4 chemistry books and 2 physics books.
   (a) How many different books can he read? What rule would we use here?
   He can read $5 + 4 + 2 = 11$ books. We use the sum rule.
   (b) How many different ways can he read 1 biology book, 1 chemistry book and 1 physics book? What rule did we use here?
   He can read one of each subject $5 \times 4 \times 2 = 40$ different ways. We use the product rule.
2. Jenny the Jeweller is trying to make a home-made necklace out of beads. All the beads are unique.
   (a) How many different ways can she make necklaces if she has 6 beads?
   She can make a necklace $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ different ways.
   (b) How many different ways can she make necklaces if she has 30 beads? **DO NOT EVALUATE**
   She can make a necklace $30 \times 29 \times 28... \times 3 \times 2 \times 1$ different ways.
3. Izzy the Ice Cream Lover goes to the UW Math Shop to get some ice cream. Izzy can pick either strawberry, chocolate or vanilla for his flavour and can choose to have it in a cup or a cone. He tells the cashier that he's upset that he can only have 5 combinations, but the cashier tells him he's wrong. Why?

   Izzy used the sum rule and added all of his options together. He can have strawberry **OR** chocolate **OR** vanilla, **AND** cone **OR** cup. Mathematically this represents $(1 + 1 + 1) \times (1 + 1) = 3 \times 2 = 6$ combinations.

Exercises II

1. Evaluate the following factorials:

   (a) $10!$

   $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3628800$

   (b) $\frac{15!}{13!}$

   $$\frac{15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 15 \times 14 = 210$$

   (c) $\frac{4!}{9!}$

   $$\frac{4 \times 3 \times 2 \times 1}{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{1}{9 \times 8 \times 7 \times 6 \times 5} = \frac{1}{15120}$$

   (d) $\frac{5! \times 3!}{8! \times 2!}$

   $$\frac{5 \times 4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 2 \times 1} = \frac{3}{8 \times 7 \times 6} = \frac{3}{8 \times 7 \times 3 \times 2} = \frac{1}{112}$$

   (e) $\frac{4! \times 6! \times 3!}{6! \times 2! \times 2! \times 5!}$

   $$\frac{4! \times 3!}{2! \times 2! \times 5!} = \frac{4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{3}{5 \times \times 2 \times 1} = \frac{3}{10}$$

Problem Set

1. Complete the following rows of Pascal's Triangle (**Hint:** You can do this by writing out the entire triangle or use the choose function to find each entry).

   (a) Complete the 9th row.

   $$\binom{9}{0}\binom{9}{1}\binom{9}{2}\binom{9}{3}\binom{9}{4}\binom{9}{5}\binom{9}{6}\binom{9}{7}\binom{9}{8}\binom{9}{9}$$

   1 9 36 84 126 126 84 36 9 1

   (b) Complete the 14th row.

   $$\binom{14}{0}\binom{14}{1}\binom{14}{2}\binom{14}{3}\binom{14}{4}\binom{14}{5}\binom{14}{6}\binom{14}{7}\binom{14}{8}\binom{14}{9}\binom{14}{10}\binom{14}{11}\binom{14}{12}\binom{14}{13}\binom{14}{14}$$

   1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1

2. Find the missing number in this row. (**Hint:** Looking at the numbers is useful, but how many entries are there?)

$$1 \quad \_ \quad 78 \quad 186 \quad 715 \quad 1287 \quad 1716 \quad 1716 \quad 1287 \quad 715 \quad 186 \quad 78 \quad \_ \quad 1$$

Starting at 0, if we count the entries, there are 13, meaning this is the 13th row.

3. How many two-digit positive numbers end in a 5 or 7?

What are the two cases?

Case 1: The number ends in a 5. The numbers have the form [X]5. There are 9 choices for the first digit (1,...,9), and one choice for the second digit, namely 5. Hence there are $9 \times 1 = 9$ two digit numbers that end in 5. Note that the first digit cannot be zero if the number is a two digit number.

Case 2: The number ends in a 7. The numbers have the form [X]7. There are 9 choices for the first digit (1,...,9), and one choice for the second digit, namely 7. Hence there are $9 \times 1 = 9$ two digit numbers that end in 7. Note that the first digit cannot be zero if the number is a two digit number.

Thus, by the sum rule, there are $9 + 9 or 18$ two digit numbers which end in a 5 or 7.

4. A restaurant from Groupon has three courses: 5 Appetizers, 4 Main Courses and 3 Desserts.

   (a) How many different meals consisting of one appetizer, one main course and one dessert could you order?

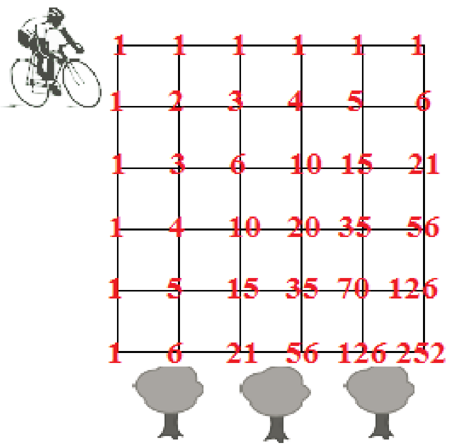
   $$5 \text{ Appetizers} \times 4 \text{ Main Courses} \times 3 \text{ Desserts} = 60 \text{ different meals.}$$

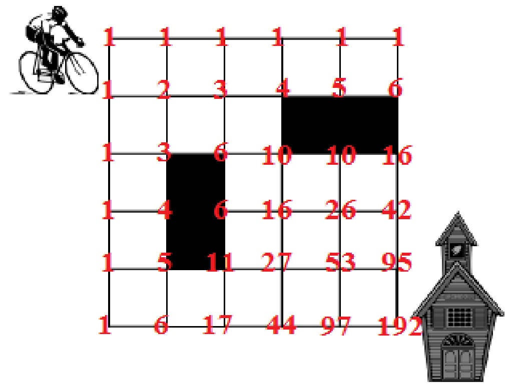
   (b) How many different meals consisting of at least one item could you order? You are not allowed to have more than one item from a particular group.

   We'll consider "No Appetizer", "No Main Course" and "No Dessert" as separate courses in each category:

   $$6 \text{ Appetizers} \times 5 \text{ Main Courses} \times 43 \text{ Desserts} = 120 \text{ different meals.}$$

5. Count how many paths James can take to get to each destination if he can only travel down or to the right. For each example we count his paths at each corner of the squares.





Left: Since the entire bottom row is James' destination we must find the sum of every path on the bottom row:

$$1 + 6 + 21 + 56 + 126 + 252 = 462 \text{ ways to get to the wooded area}$$

Right: There are 192 ways for James to get to his destination.

6. How many positive numbers less than 1000 have 1 as the first digit?

There are 3 cases: three digit numbers which have a first digit one, two digit numbers which have a first digit one and one digit numbers which have a first digit one. We will use the product rule for each case to determine the number of numbers in that particular case. Then, since there is no overlap between cases, we will use the sum rule to determine the overall number of possible numbers.

Case 1: Three digit numbers of the form 1[X][O]. The first digit must be 1, there are 10 choices for [X] and 10 choices for [O]. Hence, by the product rule, there are $1 \times 10 \times 10$ or 100 possible three digit numbers whose first digit is one.

Case 2: Two digit numbers of the form 1[X]. There are 10 choices for [X] and the first digit must be a one. Hence, by the product rule, there are 1×10 or 10 possible two digit numbers whose first digit is one.

Case 3: There is only 1 one digit number whose first digit is 1.

Using the sum rule, there are $100 + 10 + 1$ or 111 numbers less than 1000 which have 1 as the first digit.

7. A vehicle licence plate number consists of 3 letters followed by 3 digits. How many different licence plates are possible?

   There are 10 numbers and 26 letters we could use to make license plates. To make a license plate we need a letter **AND** a letter **AND** a letter **AND** a number **AND** a number **AND** a number. this results in:

   $$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17576000 \text{ license plate possibilities}$$

8. Luc, Ryan, Alysha and Vince liked Frozen so much that they are going to see it a second time, and this time they're bringing Tim. When the friends go to the theatre they all sit in a row of six seats. Assuming they do not have to sit together in the row of six and one of the six seats is left empty, how many different seating arrangements of the five friends are possible in a row with six seats?

   For each seat, the options for persons sitting in them are: Luc, Ryan, Alysha, Tim and empty. Following our previous patterns of 6 options for the first seat, 5 options for the second and so on... we get:

6	**5**	**4**	**3**	**2**	**1**

   $$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720 \text{ seating options}$$

9. The following is a portion of Pascal's Triangle. Find the values of X, Y and Z:

   1287 1716 1716 **1287**
   3003 3432 **3003**
   6435 6435
   12870

10. The whole numbers from 1 to 1000 are written. How many of these numbers have at least two 4's appearing side-by-side?

    Case 1: Three 4's beside each other. The only three-digit number that fits this is 444.

    Case 2: Two 4's beside each other:

    - Case2-1: The number is of the form 44[X]. There are 10 numbers which can replace [X]. This means there are 10 options for this sub-case.

- Case2-2: The number is of the form [X]44. There are 10 numbers which can replace [X]. This means there are 10 options for this sub-case.

Using the sum rule, there are $10 + 10 + 1$ or 21 numbers less than 1000 which have at least two 4's appearing side-by-side.

11. How many 10 digit phone numbers are possible:

    (a) How many 10 digit phone numbers are possible using digits from 0 - 9?

10	10	10	10	10	10	10	10	10	10

    $10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10000000000$ phone numbers

    (b) How many 10 digit phone numbers are possible using digits from 0 - 9 if the last digit must be even?

    There are only 5 even numbers (since we consider 0 to be even) for the last digit:

10	10	10	10	10	10	10	10	10	5

    $10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 5 = 5000000000$ phone numbers

12. Calvin Klein has two closets. In Closet1 he has 4 shirts, 7 pants and 2 pairs of shoes. In Closet2 he has 6 shirts, 3 pants and 4 pairs of shoes. How many outfits does he have:

    (a) If he can mix-and-match items from either closet?

    Since he can mix-and-match from each closet we must first add together like items and then use the product rule:

    $$(4+6) \text{ shirts} \times (7+3) \text{ pants} \times (2+4) \text{ shoes} =$$
    $$10 \times 10 \times 6 \text{ or } 600 \text{ outfit combinations.}$$

    (b) If he cannot mix-and-match items from the closets?

    Since he cannot mix-and-match from each closet we must first multiply together each closet's items and then use the sum rule:

    $$(4 \times 7 \times 2) \text{ outfits in Closet 1} + (6 \times 3 \times 4) \text{ outfits in Closet 2} =$$
    $$56 + 72 = 128 \text{ outfit combinations.}$$

13. (a) If Emily, Nadine and Joanne were running a race, how many different orders of the Top 3 can there be?

There are three runners who could take first place, then two runners who could take second place and the last runner ends up in third place:

$$3 \times 2 \times 1 = 6 \text{ orders}$$

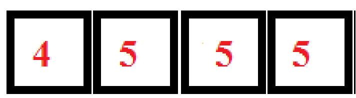
(b) if Mike, Jon, Terry and Frank were running a race, how many ways can a Top 3 be made.

There are four runners who could take first place, then three runners who could take second place and two runners who could end up in third place:

$$4 \times 3 \times 2 = 24 \text{ orders}$$

14. ** How many numbers between 1000 and 9999 have only even digits?

We only have five numbers that we can consider even (0, 2, 4, 6 and 8). For the first digit, we cannot have 0, as that would result in a number that is below 1000. Therefore there are four options for the first digit, and five optiosn for every digit after that. We then get:

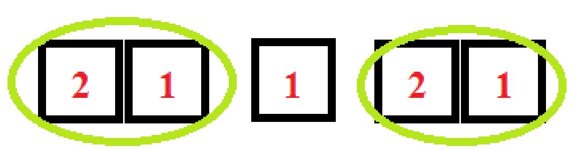


$$4 \times 5 \times 5 \times 5 = 500 \text{ numbers}$$

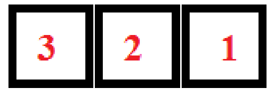
15. ** Luc, Ryan, Alysha, Vince and Tim are addicted to Frozen at this point. They are seeing it a third time. Friendships are wearing thin and Ryan said he wants to sit beside Vince, Alysha was okay with this as long as she got to sit beside Luc. Nobody really likes Tim, so they don't care if they sit beside him or not. How many different ways can the five friends sit together...

(a) ...in five seats?

For each seat, the options for persons sitting in them are: Luc-Alysha, Ryan-Vince and Tim. This only leaves us with 3 items. In the pairings, there are two ways the first friend can have a seat and one way the second friend can have a seat:



Now if we consider the question talking about just these 3 items (Tim, Vince-Ryan and Alysha-Luc) we get:

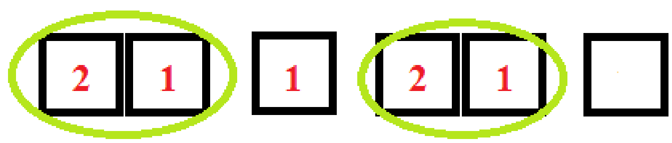


Putting all this together we get three ways for the first item, two ways for the second item and one way for the last item. However, the Vince-Ryan had two different orders and Alysha-Luc has two different orders:

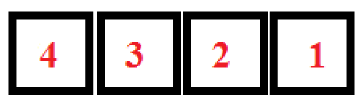
$$3 \times 2 \times 1 \times 2 \times 2 = 24 \text{ orders}$$

(b) ...in six seats?

For each seat, the options for persons sitting in them are: Luc-Alysha, Ryan-Vince, Tim and empty. This only leaves us with 4 items. In the pairings, there are two ways the first friend can have a seat and one way the second friend can have a seat:



Now if we consider the question talking about just these 4 items (Tim, Vince-Ryan, Alysha-Luc and empty) we get:



Putting all this together we get four ways for the first item, three ways for the second item, two ways for the third item and one way for the last item. However, the Vince-Ryan had two different orders and Alysha-Luc has two different orders:

$$4 \times 3 \times 2 \times 1 \times 2 \times 2 = 96 \text{ orders}$$

16. **** Show that $\binom{12}{2} + \binom{12}{3} = \binom{13}{3}$ by using the choose formula to re-write the equation.

$$\binom{12}{2} + \binom{12}{3} = \frac{12!}{2!(12-2)!} + \frac{12!}{3!(12-3)!}$$
$$\binom{12}{2} + \binom{12}{3} = \frac{12!}{2!(10)!} + \frac{12!}{3!(9)!}$$
$$\binom{12}{2} + \binom{12}{3} = \frac{12 \times 11}{2 \times 1} + \frac{12 \times 11 \times 10}{3 \times 2 \times 1}$$
$$\binom{12}{2} + \binom{12}{3} = \frac{12 \times 11 \times 3}{3 \times 2 \times 1} + \frac{12 \times 11 \times 10}{3 \times 2 \times 1}$$
$$\binom{12}{2} + \binom{12}{3} = \frac{396}{6} + \frac{1320}{6}$$
$$\binom{12}{2} + \binom{12}{3} = \frac{1716}{6}$$
$$\binom{12}{2} + \binom{12}{3} = 286$$

Now let's check $\binom{13}{3}$:

$$\binom{13}{3} = \frac{13!}{3!(13-3)!}$$
$$\binom{13}{3} = \frac{13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$
$$\binom{13}{3} = \frac{13 \times 12 \times 11}{3 \times 2 \times 1}$$
$$\binom{13}{3} = \frac{1716}{6}$$
$$\binom{13}{3} = 286$$

Therefore: $\binom{12}{2} + \binom{12}{3} = \binom{13}{3}$