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Grade 7/8 Math Circles

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Counting I

Counting

Counting in mathematics is determining the number of ways in which something can occur. The study of the method of counting is called **combinatorics**. First we must learn the building blocks to solving more difficult questions we may come across.

Example: Ryan wants to get dressed in the morning for work. He has 2 different hats, 3 different bottoms and 2 different tops. How many different combinations of outfits could he have?

Solution: Let's write out the entire set of combinations:

- where h represents the two different hats
- where t represents the two different tops
- where b represents the three different bottoms

$h_1\ t_1\ b_1$	$h_1\ t_2\ b_1$	$h_2\ t_1\ b_1$	$h_2\ t_2\ b_1$
$h_1\ t_1\ b_2$	$h_1\ t_2\ b_2$	$h_2\ t_1\ b_2$	$h_2\ t_2\ b_2$
$h_1\ t_1\ b_3$	$h_1\ t_2\ b_3$	$h_2\ t_1\ b_3$	$h_2\ t_2\ b_3$

This was a relatively easy task to do due to Ryan's lack of clothing. However, what would happen if he had 32 hats, 47 tops and 119 bottoms? It's probably much more difficult to realize that the answer is a total of 178976 combinations.

There are counting rules that we will explore to help make larger problems like this easier.

Counting Rules

A common technique in problem solving is to separate a more difficult question into smaller, more manageable problems. Two counting techniques which aid us in this are the **Rule of Product** and the **Rule of Sum**.

In our lesson on sets we learned to think of "*AND*" and "*OR*" differently to help us solve problems. With counting techniques, it is necessary to think of **"*AND*" as multiplication** and **"*OR*" as addition**. Of course questions do not always have an AND or an OR in them and you may have to play around with re-wording the question to discover the implied ANDs or ORs.

Product Rule (Fundamental Counting Principle)

The **Product Rule** states:

> *If Action 1 can be done in p ways and, **for each of these ways**, we can do Action 2 in q ways, then we can do both Action 1 **AND** Action 2 in $p \times q$ ways.*

The Product Rule holds true for any number of actions.

Example: A company makes both 1" and 2" binders in one of four colours: red, green, blue, black. How many different binders can they make in total?

> **Solution:** We can re-word this question as: *1. "How many binders can we make if we have 2 sizes **AND** 4 colours?".*
>
> 1. 2 sizes $\times$ 4 colours $=$ 8 binders in total.

Sum Rule

The **Sum Rule** state:

> *If Action 1 can be done in p ways and Action 2 can be done in q ways, then we can do both Action 1 **OR** Action 2 in $p + q$ ways.*

The Sum Rule holds true for any number of actions.

Example: A company makes both 1" and 2" binders in one of four colours: red, green, blue, black. How many different binders can they make in total?

> **Solution:** We can re-word this question in two ways: *1. "How many 1"* ***OR*** *2" binders are there?"* or *2. "How many red* ***OR*** *green* ***OR*** *blue* ***OR*** *black binders are there?"*.
>
> 1. 4 - 1" binders + 4 - 2" binders = 8 binders in total.
> 2. 2 red binders + 2 green binders + 2 blue binders + 2 black binders = 8 binders in total.

Grouping

Many problems involve creating groups of objects and counting how many ways we can create smaller groups out of a larger set.

Example: Friends Vince, Ryan, Alysha, and Luc go to the movie theatre to see Frozen. All four friends sit together in the same row with 4 seats.

(a) How many different seating arrangements of the four friends are possible?

Solution: If we draw out the four seats.

Any one of the 4 friends could sit in the first seat AND THEN any one of the remaining 3 could sit in the second seat AND THEN there are 2 friends who could sit in the third seat AND THEN the fourth seat goes to the friend who does not have a seat.

4	3	2	1

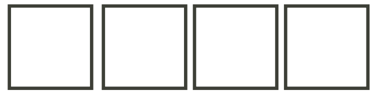
Now by applying the fundamental counting principle we have $4 \times 3 \times 2 \times 1 = 24$.

Therefore there are 24 possible seating arrangements for the friends at the theatre.

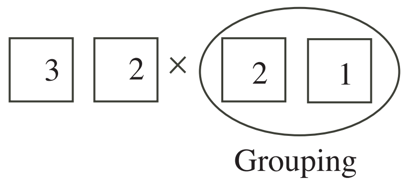
(b) Vince and Ryan are BFFLs. They want to sit together for the movie. How many ways can they sit together in the same row with 4 seats?

These questions, while seemingly difficult, can be made easier by separating the problem into smaller pieces. By solving the smaller sub-groups through regular counting rules, and then counting this entire sub-group as an object, we can now solve the problem overall.

Solution: If we draw out the four seats.



Since Ryan and Vince are BFFLs, we can consider them a single item. Within that smaller group, there are two seats the first can take and one that the second can take. That means this smaller grouping has a value of: $2 \times 1 = 2$ possibilities.



Now if we look at the seats as needing to hold Luc, Alysha and the Vince/Ryan-pair, we only have 3 seats to work with. We see there are 3 seats for the first friend, 2 seats for the second friend and 1 seat for the last friend.

BUT remember that one of our "friends" is a pair now and has a value of 2, so we end up with: $3 \times 2 \times 2 = 12$.

Exercises I

1. Barry the Bookworm has 5 biology books, 4 chemistry books and 2 physics books.
   (a) How many different books can he read? What rule would we use here?
   (b) How many different ways can he read 1 biology book, 1 chemistry book and 1 physics book? What rule did we use here?

2. Jenny the Jeweller is trying to make a home-made necklace out of beads. All the beads are unique.
   (a) How many different ways can she make necklaces if she has 6 beads?
   (b) How many different ways can she make necklaces if she has 30 beads? **DO NOT EVALUATE**

3. Izzy the Ice Cream Lover goes to the UW Math Shop to get some ice cream. Izzy can pick either strawberry, chocolate or vanilla for his flavour and can choose to have it in a cup or a cone. He tells the cashier that he's upset that he can only have 5 combinations, but the cashier tells him he's wrong. Why?

Factorials

One the most important uses of factorials are problems in which total outcomes must be determined. We saw this in 2b where we had to write out a long multiplication statement. We simply needed to think in terms of how many options there were to order the beads. This was a generally straightforward example because there were only 6 beads. 30 was even more difficult but what if she had 100 beads, we'd have to write out:

$$100 \times 99 \times 98 \times 97 \times 96 \times ... \times 5 \times 4 \times 3 \times 2 \times 1$$

But there is an easier way that can summarize this set of multiplication statements. We just have to write 100!.

A common mathematical notation in counting is the **factorial notation**. The factorial of some positive integer, written as $n!$, is represented as:

$$(n) \times (n-1) \times (n-2) \times ... \times (2) \times (1)$$

Let's clear this up with an example. We can show that:

$$5! \; = \; 5 \times 4 \times 3 \times 2 \times 1 \; = \; 120$$

Essentially all we need to do is multiply every number from 1 to n to find n!. But factorials have some restrictions and special cases that we must look at:

- 0! is a special case that we must remember. We can say that $0! = 1$.
- We can only find the factorial of integers (whole numbers). For example 3.14! or π! is not possible and therefore has no answer.
- We cannot find the factorial of negative integers. Therefore we cannot get an answer for $(-5)!$.

Multiplying & Dividing Factorials

While dividing factorials may seem like an easy task if you have a calculator on-hand, if your calculator does not have a factorial command, manually finding it by typing out expanded factorials can be a tedious task, especially with larger numbers. Learning to simplify factorials when we divide can make working with them much easier. Let's look at two cases:

Dividing: Numerator is Larger

After we have expanded all of the factorials in an expression, we can look for any common numbers. If the numerator is larger than the denominator we can see that there will be a product remaining in the numerator, but the denominator will be completely cancelled out.

Example: Simplify $\frac{6!}{3!}$

Solution: Without using our calculator and expanding the factorials we get:

$$\frac{6\times5\times4\times3\times2\times1}{3\times2\times1}$$

We can see that in both the numerator and denominator, there exists a "$3\times2\times1$". So let's cancel them out. This leaves us with:

$$6 \times 5 \times 4 = 120$$

Dividing: Denominator is Larger

After we have expanded all of the factorials in an expression, we can look for any common numbers. If the numerator is larger we can see that there will be a product remaining in the numerator.

If the denominator is larger than the numerator we can see that there will be a product remaining in the denominator, but the numerator will be completely cancelled out. If this occurs and nothing remains in the numerator, we must place a "1" in the numerator as a place-holder.

Example: Simplify $\frac{5!}{7!}$

Solution: Without using our calculator and expanding the factorials we get:

$$\frac{5\times4\times3\times2\times1}{7\times6\times5\times4\times3\times2\times1}$$

We can see that in both the numerator and denominator, there exists a "$5 \times 4 \times 3 \times 2 \times 1$". So let's cancel them out. Remember, because everything in the numerator is gone now, we have to place a "1" in the numerator. This leaves us with:

$$\frac{1}{7\times6} = \frac{1}{42}$$

Multiplying: Either Scenario

Multiplying factorials together does not introduce any options for us to simplify the expression. However, when we introduce multiplication together with division, it requires us to write out the expanded form of the factorials to assure that we cancel out appropriately. Remembering that larger parts of the factorials may further be factored allows for further simplifying of the expression.

Example: Simplify $\frac{6!\times 3!}{8!}$

Solution: Without using our calculator and expanding the factorials we get:

$$\frac{6\times 5\times 4\times 3\times 2\times 1\times 3\times 2\times 1}{8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}$$

We can see that in both the numerator and denominator, there exists a "$6 \times 5 \times 4 \times 3 \times 2 \times 1$". So let's cancel them out. This leaves us with:

$$\frac{3\times 2\times 1}{8\times 7}$$

We may think this is as far as we can go. But we know that $8 = 2 \times 2 \times 2$. So let's insert this into our example:

$$\frac{3\times 2\times 1}{2\times 2\times 2\times 7}$$

Let's cancel out a 2 from the numerator and denominator:

$$\frac{3\times 1}{2\times 2\times 7} = \frac{3}{28}$$

Exercises II

1. Evaluate the following factorials:

   (a) $10!$

   (b) $\frac{15!}{13!}$

   (c) $\frac{4!}{9!}$

   (d) $\frac{5! \times 3!}{8! \times 2!}$

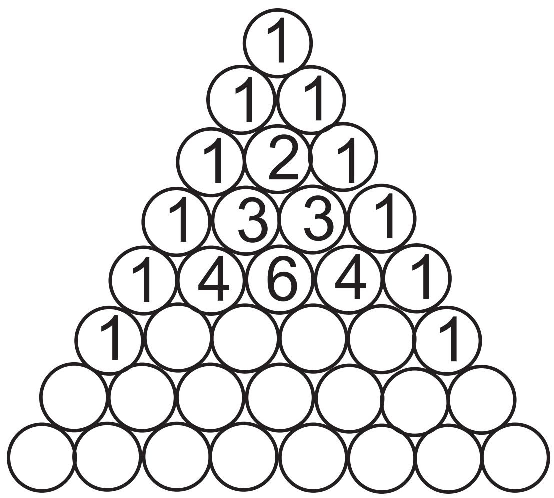
   (e) $\frac{4! \times 6! \times 3!}{6! \times 2! \times 2! \times 5!}$

Pascal’s Triangle

Pascal’s Triangle is an interesting number pattern named after Blaise Pascal, a famous French mathematician. It has many uses in counting paths and it’s use in the combination function will become really important to us for our next lesson.

Building the Pascal’s Triangle

To build the triangle we start with 1 at the top, and continue adding numbers in a triangular shape. The leftmost and rightmost diagonals of Pascal’s Triangle are 1s, and each number in between is the sum of the two numbers above it.



Rows and Elements

Pascal’s Triangle has a unique classification method in order to identify specific numbers in the triangle. We can classify a number based on it’s row and entry:

- A **row** refers to the horizontal set of numbers in the Pascal’s Triangle. We count the very top “1” as being a part of Row 0. The counting scheme continues as normal and you can easily determine the row number by looking at the second number in the row (or the first non-1 value).
- An **entry** refers to a specific number in a said row. We call the very first number in a row as Entry 0 (which will always be the number 1). The counting scheme continues as normal, and the number of entries for a row are always between 0 and the row’s number. For example: Row 3 {1331} has entry numbers 0, 1, 2, 3.

Combinations and the Triangle: Finding any Entry

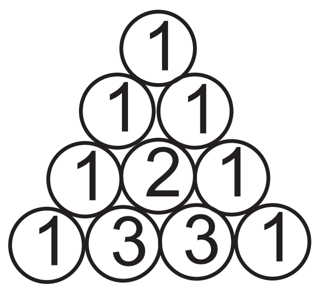
The formula for any entry in Pascal's Triangle can be found using the combination formula. This formula is just the choose function which we will be using a lot next week and for probability. Right now, let's focus on using it to find entries for Pascal's Triangle, but remember its use when we look into combinations. The formula is:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

(where n is the row number, and k is the entry such that $0 \leq k \leq n$*)*

Let's see what this looks like when we compare the triangles:

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \; \binom{1}{1} \\
\binom{2}{0} \; \binom{2}{1} \; \binom{2}{2} \\
\binom{3}{0} \; \binom{3}{1} \; \binom{3}{2} \; \binom{3}{3}
\end{array}$$



Example: Find the number for each entry:

(a) 6th entry of the 8th row.

$$\binom{8}{6} = \frac{8!}{6!(8-6)!}$$
$$\binom{8}{6} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 2 \times 1}$$
$$\binom{8}{6} = \frac{2 \times 2 \times 2 \times 7}{2 \times 1}$$
$$\binom{8}{6} = \frac{2 \times 2 \times 7}{1}$$
$$\binom{8}{6} = 28$$

(b) 12th entry of the 10th row.

We cannot find this because the 10th row only has entries from 0 to 10. It does not have a 12th entry.

Problem Set

1. Complete the following rows of Pascal's Triangle (**Hint:** You can do this by writing out the entire triangle or use the choose function to find each entry).
   (a) Complete the 9th row.
   (b) Complete the 14th row.

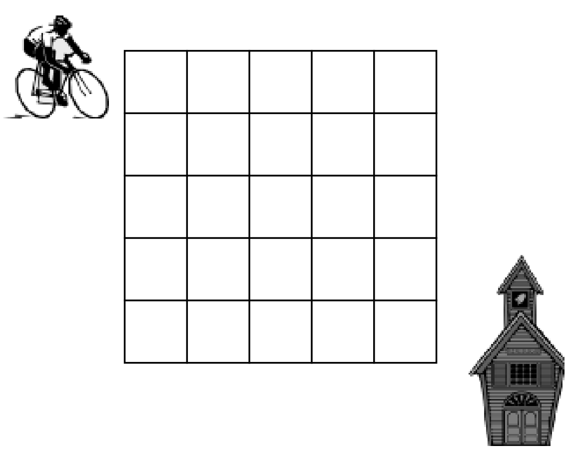
2. Find the missing number in this row. (**Hint:** Looking at the numbers is useful, but how many entries are there?)

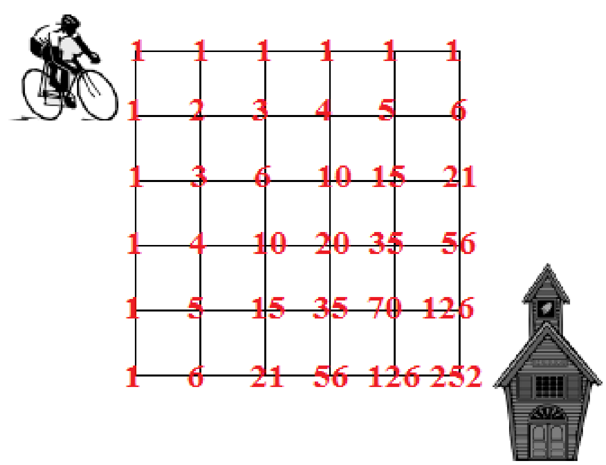
   1 _ 78 186 715 1287 1716 1716 1287 715 186 78 _ 1

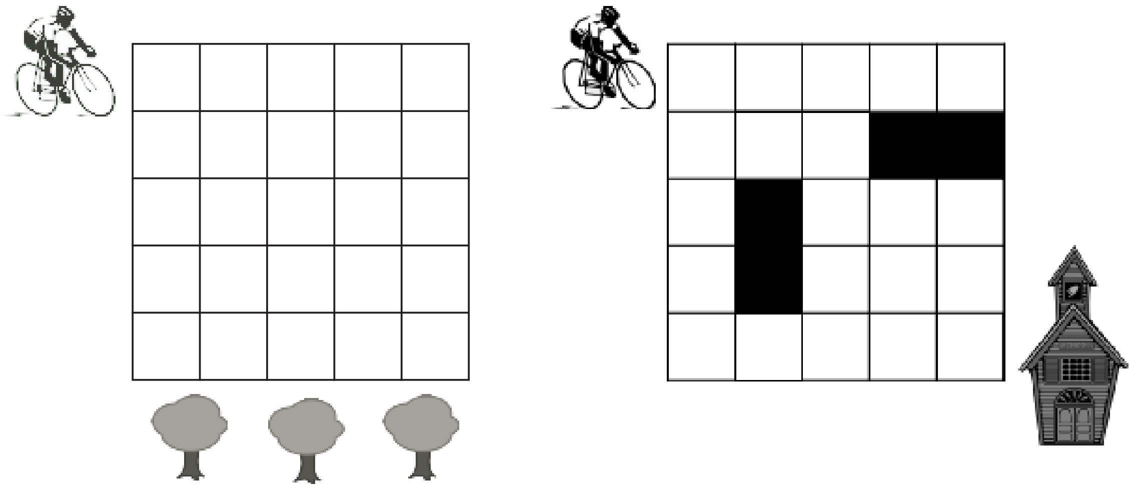
3. How many two-digit positive numbers end in a 5 or 7?

4. A restaurant from Groupon has three courses: 5 Appetizers, 4 Main Courses and 3 Desserts.
   (a) How many different meals consisting of one appetizer, one main course and one dessert could you order?
   (b) How many different meals consisting of at least one item could you order? You are not allowed to have more than one item from a particular group.

5. Count how many paths James can take to get to each destination if he can only travel down or to the right. For each example we count his paths at each corner of the squares. Here's an example:







Hint: In the left example, James considers the entire bottom line his destination. In the right example, James cannot cross blacked out squares.

6. How many positive numbers less than 1000 have 1 as the first digit?

7. A vehicle licence plate number consists of 3 letters followed by 3 digits. How many different licence plates are possible?

8. Luc, Ryan, Alysha and Vince liked Frozen so much that they are going to see it a second time, and this time they're bringing Tim. When the friends go to the theatre they all sit in a row of six seats. Assuming they do not have to sit together in the row of six and one of the six seats is left empty, how many different seating arrangements of the five friends are possible in a row with six seats?

9. The following is a portion of Pascal's Triangle. Find the values of X, Y and Z:

$$\begin{array}{ccccccc} 1287 & & \_\_\_\_ & & \_\_\_\_ & & X \\ & 3003 & & 3432 & & Y & \\ & & \_\_\_\_ & & 6435 & & \\ & & & Z & & & \end{array}$$

10. The whole numbers from 1 to 1000 are written. How many of these numbers have at least two 4's appearing side-by-side?

11. How many 10 digit phone numbers are possible:

    (a) How many 10 digit phone numbers are possible using digits from 0 - 9?

(b) How many 10 digit phone numbers are possible using digits from 0 - 9 if the last digit must be even?

12. Calvin Klein has two closets. In Closet1 he has 4 shirts, 7 pants and 2 pairs of shoes. In Closet2 he has 6 shirts, 3 pants and 4 pairs of shoes. How many outfits does he have:

    (a) If he can mix-and-match items from either closet?

    (b) If he cannot mix-and-match items from the closets?

13. (a) If Emily, Nadine and Joanne were running a race, how many different orders of the Top 3 can there be?

    (b) if Mike, Jon, Terry and Frank were running a race, how many ways can a Top 3 be made.

14. ** How many numbers between 1000 and 9999 have only even digits?

15. ** Luc, Ryan, Alysha, Vince and Tim are addicted to Frozen at this point. They are seeing it a third time. Friendships are wearing thin and Ryan said he wants to sit beside Vince, Alysha was okay with this as long as she got to sit beside Luc. Nobody really likes Tim, so they don't care if they sit beside him or not. How many different ways can the five friends sit together...

    (a) ...in five seats?

    (b) ...in six seats?

16. **** Show that $\binom{12}{2} + \binom{12}{3} = \binom{13}{3}$ by using the choose formula to re-write the equation.