UNIVERSITY OF
WATERLOO
Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles
February 18/19, 2014
Counting II - Solutions

Exercises I

1. For each of the following scenarios, state whether order matters or not:

   (a) The number of ways three distinct plants can be arranged on a window sill.

   ORDER DOES MATTER

   (b) Selecting a combination on a combination lock.

   ORDER DOES MATTER

   (c) A math student is given a list of 5 math problems and is asked to solve any 3 of the problems. How many different problem selections can the student make?

   ORDER DOESN'T MATTER

   (d) Mr E. Lipps has in front of him a circle, a square, and a rectangle. In how many ways can he select some of the shapes? (He may pick any number of the shapes)

   ORDER DOESN'T MATTER

   (e) How many permutations are there of the numbers 1 - 5 taken 2 at a time? List them.

   ORDER DOES MATTER

   (f) How many different ways can the letters in STATISTICS be arranged?

   ORDER DOES MATTER

Exercises II

1. State if the question requires Basic Permutations (PB), K-Permutations of N (PK) or Permutations with Repetitions (PR) to solve:

   (a) The number of ways three distinct plants can be arranged on a window sill.

   Basic Permutations

   (b) Selecting a combination on a spinwheel combination lock.

   K-Permutations of N OR Permutations with Repetitions

   (c) How many permutations are there of the numbers 1 - 5 taken 2 at a time. List them.

   K-Permutations of N

   (d) How many different ways can the letters in CALCULATOR be arranged?

   Permutations with Repetitions

   (e) How many ways can Jenny the Jeweller make a bracelet with 30 distinct beads?

   Basic Permutations

Problem Set

1. Given the set { 1, 2, 3, 4 }:

   (a) How many permutations are there on the set? List them.

1 2 3 4	1 2 4 3	1 3 2 4	1 3 4 2	1 4 2 3	1 4 3 2
2 1 3 4	2 1 4 3	2 3 1 4	2 3 4 1	2 4 1 3	2 4 3 1
3 1 2 4	3 1 4 2	3 2 1 4	3 2 4 1	3 4 1 2	3 4 2 1
4 1 2 3	4 1 3 2	4 2 1 3	4 2 3 1	4 3 1 2	4 3 2 1

   (b) How many permutation on the set start with 2? List them.

2 1 3 4	2 1 4 3	2 3 1 4	2 3 4 1	2 4 1 3	2 4 3 1

   (c) How many permutations on the set have 1 and 2 beside each other? List them.

1 2 3 4	1 2 4 3	2 1 3 4	2 1 4 3

2. Evaluate the following:

(a) $\binom{5}{3}$

$$\begin{aligned}\binom{5}{3} &= \frac{5!}{3!(5-3)!} \\ &= \frac{5\times 4\times 3\times 2\times 1}{3\times 2\times 1\times 2\times 1} \\ &= \frac{20}{2} \\ &= 10\end{aligned}$$

(b) $\binom{10}{6}$

$$\begin{aligned}\binom{10}{6} &= \frac{10!}{6!(10-6)!} \\ &= \frac{10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{6\times 5\times 4\times 3\times 2\times 1\times 4\times 3\times 2\times 1} \\ &= \frac{5040}{24} \\ &= 210\end{aligned}$$

(c) $\dfrac{\binom{7}{6}}{\binom{4}{2}}$

$$\begin{aligned}\frac{\binom{7}{6}}{\binom{4}{2}} &= \frac{7!}{6!1!} \div \frac{4!}{2!2!} \\ &= \frac{7!}{6!1!} \times \frac{2!2!}{4!} \\ &= \frac{7\times 6\times 5\times 4\times 3\times 2\times 1\times 2\times 1\times 2\times 1}{6\times 5\times 4\times 3\times 2\times 1\times 1\times 4\times 3\times 2\times 1} \\ &= \frac{7\times 2\times 1}{4\times 3} \\ &= \frac{14}{12} = \frac{7}{6}\end{aligned}$$

3. A student club with 10 members wishes to select a president, a secretary and a treasurer from its membership. No member may be selected for more than 1 office. In how many ways can this be done?

There are 10 members and we want to choose 3:

$$\begin{aligned}\binom{10}{3} &= \frac{10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{7\times 6\times 5\times 4\times 3\times 2\times 1\times 3\times 2\times 1} \\ &= \frac{10\times 9\times 8}{3\times 2\times 1} \\ &= \frac{720}{6} = 120\end{aligned}$$

4. Calculate the number of permutations on a set with 3 A's, 6 B's and 3 C's.

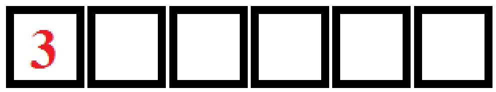
We have $(3+6+3 = 12$ letters to work with. We have a repeat of 3, a repeat of 6 and a repeat of 3:

$$\frac{12!}{3!6!3!} = \frac{479001600}{25920} = 18480$$

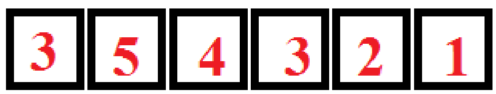
5. How many permutations of the numbers { 1, 2, 3, 4, 5, 6 }:

   (a) begin with an even number?

   We must start off with one of the 3 even numbers:



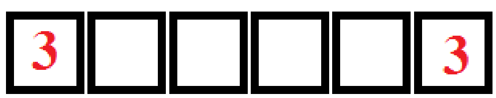
   Now we have 5 items left and can continue using our basic permutation:



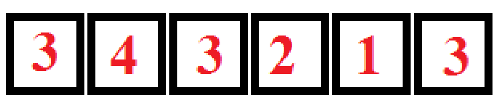
   We now have $3 \times 5 \times 4 \times 3 \times 2 \times 1 = 320$ combinations.

   (b) begin with an odd number and end with an even number?

   We must start off with one of the 3 odd numbers and finish off with one of the 3 even numbers:



   Now we have 4 items left and can continue using our basic permutation:



   We now have $3 \times 4 \times 3 \times 2 \times 1 \times 3 = 216$ combinations.

(c) begin with an even number and end with an even number?

We must start off with one of the 3 even numbers:

3					

After getting rid of one even number, we have 2 for the end:

3					2

Now we have 4 items left and can continue using our basic permutation:

3	4	3	2	1	2

We now have $3 \times 4 \times 3 \times 2 \times 1 \times 2 = 144$ combinations.

6. A school has 380 female students and 120 male students. They must create a 5-person student council.

(a) In how many ways can they do this?

$$\binom{380+120}{5} = \binom{500}{5} = \frac{500!}{5!495!} = \frac{500 \times 499 \ ... \ \times 2 \times 1}{5 \times ... \times 1 \times 495 \times ... \times 1}$$

$$= \frac{500 \times 499 \times 498 \times 497 \times 496}{5 \times 4 \times 3 \times 2 \times 1} \quad \text{OR} \quad 2.55 \times 10^{11}$$

(b) In how many ways can they do this if it can only be made of girls?

$$\binom{380}{5} = \frac{380!}{5!375!} = \frac{380 \times 379 \ ... \ \times 2 \times 1}{5 \times ... \times 1 \times 375 \times ... \times 1}$$

$$= \frac{380 \times 379 \times 378 \times 377 \times 376}{5 \times 4 \times 3 \times 2 \times 1} \quad \text{OR} \quad 6.43 \times 10^{10}$$

(c) In how many ways can they do this if there must be 4 boys and 1 girl?

$$\binom{120}{4} \times \binom{380}{1} = \frac{380!}{1!379!} \times \frac{120!}{4!116!} =$$

$$\frac{380 \times 379 \times ... \times 1}{379 \times ... \times 1} \times \frac{120 \times ... \times 1}{4 \times ... \times 1 \times 116 \times ... \times 1}$$

$$= 380 \times \frac{120 \times 119 \times 118 \times 117}{4 \times 3 \times 2 \times 1} \quad \text{OR} \quad 380 \times 8214570 = 3121536600$$

(d) In how many ways can they do this if there must be more girls than boys?

$$\binom{380}{5}\binom{120}{0} + \binom{380}{4}\binom{120}{1} + \binom{380}{3}\binom{120}{2}$$

7. Brad the Bachelor has dwindled down his choices to 10 lucky bachelorettes. How many combinations are there:

(a) If the show is getting cancelled and he needs to pick 4 finalists?

$$\begin{aligned}\binom{10}{4} &= \frac{10!}{4!(10-4)!} \\ &= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \\ &= \frac{5040}{24} \\ &= 210\end{aligned}$$

(b) If the show says he must pick Beautiful Betty and 3 other finalists?

$$\begin{aligned}\binom{9}{3} &= \frac{9!}{3!(9-3)!} \\ &= \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \\ &= \frac{504}{6} \\ &= 84\end{aligned}$$

(c) If he doesn't want to pick Smelly Sandra and must pick 4 finalists.

$$\begin{aligned}\binom{9}{4} &= \frac{9!}{4!(9-4)!} \\ &= \frac{9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{4\times 3\times 2\times 1\times 5\times 4\times 3\times 2\times 1} \\ &= \frac{3024}{24} \\ &= 126\end{aligned}$$

8. Given the set { A, A, A, B, B }:

(a) Write out all the combinations.

$BBAAA$	$BABAA$	$BAABA$	$BAAAB$	$AAABB$
$ABBAA$	$ABABA$	$ABAAB$	$AABBA$	$AABAB$

(b) Use the required formula to calculate the number of combinations and make sure that you were correct.

$$\begin{aligned}\frac{5!}{3!2!} &= \frac{5\times 4\times 3\times 2\times 1}{3\times 2\times 1\times 2\times 1} \\ &= \frac{5\times 4}{2\times 1} \\ &= \frac{20}{2} \\ &= 10\end{aligned}$$

9. * Tim the Math Circles Assistant bought samosas, but didn't give Ryan any. Ryan's just going to steal them. How many ways can he steal some of Tim's samosas (**Hint:** He can steal any number of samosas):

(a) If Tim has 3 samosas?

$$\binom{3}{1}+\binom{3}{2}+\binom{3}{3} = 3+3+1 = 7 = 2^3-1 \text{ different ways.}$$

(b) If Tim has 4 samosas?

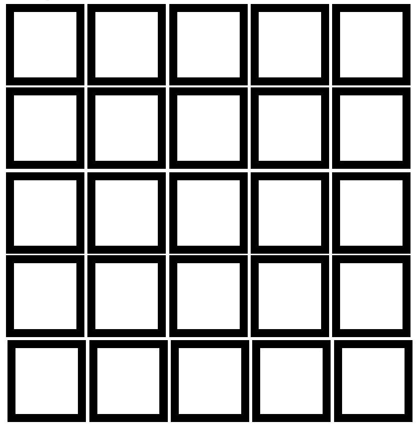
$$\binom{4}{1}+\binom{4}{2}+\binom{4}{3}+\binom{4}{4} = 4+6+4+1 = 15 = 2^4-1 \text{ different ways.}$$

(c) If Tim has n samosas?

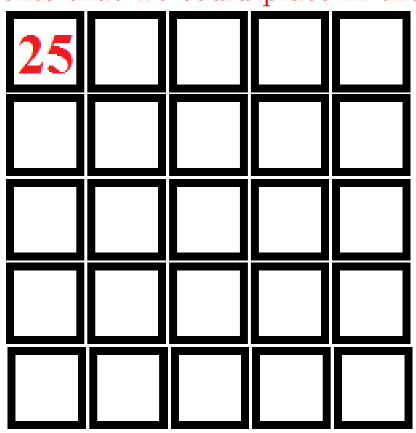
$$\binom{n}{1}+...+\binom{n}{n} = n+...+1 = 2^n-1 \text{ different ways.}$$

10. * In how many ways can 25 students be seated in a classroom with 25 desks? With 30 desks?

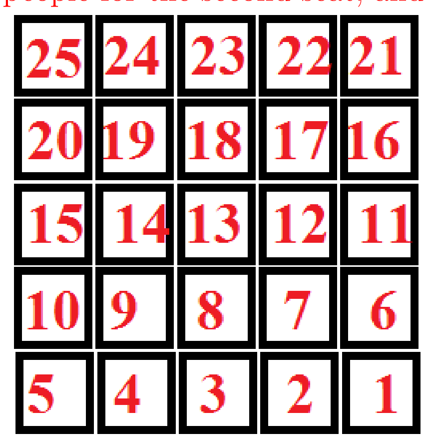
We must start off with drawing out the 25 desks.



After this there are 25 students that we could place in the first seat.



If we continue, we have 24 people for the second seat, and 23 for the third and so on..



We now have 25! different ways for the students to sit in the 25 seats.

If we had 30 seats and the 25 students we have two scenarios:

(a) How many ways can we choose the 25 seats that the students sit in?

(b) How many ways can we then permute the 25 seats that were chosen?

The two answers we get must be multiplied together because we need to know how to do both jobs. We want a AND b to occur.

(a) $\binom{30}{5} = \frac{30!}{5!(30-25)!} = \frac{30 \times ... \times 1}{5 \times ... \times 1 \times 25 \times ... \times 1} = \frac{30 \times 29 \times 28 \times 27 \times 26}{5 \times 4 \times 3 \times 2 \times 1} = \frac{17100720}{120} = 142506$

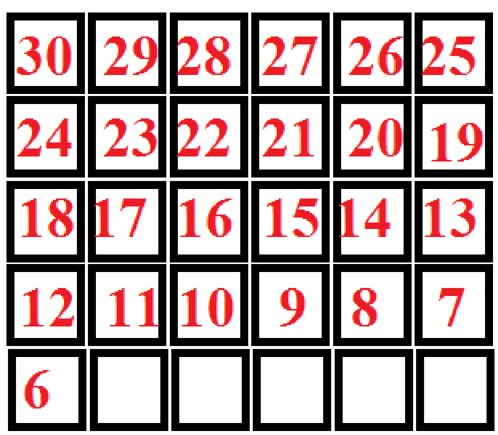
(b) The 25 seats can be permuted with 25!

We must now multiply these together to get:

$$a \times b = 142506 \times 25! = 2.21 \times 10^{30}$$

There are then approximately 2.21×10^{30} ways to choose a seat with 25 students and 30 seats.

This is essentially a K-Permutations of N problem. We must find 25-Permutations of 30. If we look at the picture, we see there are 30 seats for the first student to pick, then 29 for the second and so on... BUT since we have only 25 students, we see that the last student has 6 choices:



Mathematically we can present this as:

$$\frac{30!}{(30-25)!} = \frac{30!}{5!} = 30 \times 29 \times 28 \times ... \times 3 \times 2 \times 1 = 2.21 \times 10^{30}$$

11. * * Izzy has returned from last week for more ice cream. He has k choices of flavours and can have pick n scoops. He does not need to have different flavours for his scoops, he can have multiples of one kind. How many combinations of ice cream can he have in each case? (Listing them may help)

For the following questions we can use the stars and bars formula: $\binom{n+k-1}{k}$

For example: If he had to choose between Chocolate and Vanilla and could take 2 scoops: he could take Vanilla-Vanilla, Chocolate-Vanilla or Chocolate-Chocolate, giving 3 total combinations.

(a) • Ice Cream: Vanilla, Strawberry and Chocolate
• Scoops Allowed: 2

VV	CC	SS
VC	VS	CS

OR $\binom{3+2-1}{2} = \binom{4}{2} = 6$

(b) • Ice Cream: Vanilla, Strawberry and Chocolate
• Scoops Allowed: 3

VVV	CCC	SSS	VSS	CSS
VCC	SCC	CVV	SVV	CSV

OR

$$\binom{3+3-1}{3} = \binom{5}{3} = 10$$

(c) • Ice Cream: Vanilla, Strawberry, Green Mint and Chocolate
• Scoops Allowed: 3

VVV	CCC	SSS	GGG	VCC
VSS	VGG	CVV	CSS	CGG
SVV	SCC	SGG	GVV	GCC
GSS	VCS	VCG	VSG	CSG

OR

$$\binom{4+3-1}{3} = \binom{6}{3} = 20$$