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Grade 7/8 Math Circles

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Counting II

Counting

We have explored using counting techniques to solve basic counting problems. However, sometimes we are not concerned with how many ways something can occur but more so how we can rearrange our outcomes, and if their rearrangement is the same or a completely new result. Let's investigate this with a problem:

Example 1: Ryan, Tim, Vince and Luc are in the finals for a track meet at the UW finals. List all the possible ways that the top 3 can be made:

Solution: Let's write out the entire set of combinations for our top 3:

$R\ L\ V$	$R\ V\ L$	$R\ L\ T$	$R\ T\ L$	$R\ T\ V$	$R\ V\ T$
$L\ R\ V$	$L\ V\ R$	$L\ V\ T$	$L\ T\ V$	$L\ T\ R$	$L\ R\ T$
$V\ R\ T$	$V\ T\ R$	$V\ L\ T$	$V\ T\ L$	$V\ L\ R$	$V\ R\ L$
$T\ R\ V$	$T\ V\ R$	$T\ L\ R$	$T\ R\ L$	$T\ L\ V$	$T\ V\ L$

There are therefore 24 different scenarios for the final outcome of the race.

Example 2: A national track team wants to make a team from Ryan, Tim, Vince and Luc at UW. How many different three-man teams can be made?

Solution: Let's write out the entire set of team we could make using the four athletes:

$R\ L\ V$	$R\ L\ T$	$R\ V\ T$	$V\ T\ L$

There are therefore 4 teams that could be made of only three athletes.

You may not have known it but above we actually used two mathematical ideas known as **Permutations** and **Combinations**. When we look at the question though, in both cases we were taking three students out of a set of four. So why were our answers different?

Order

In the two questions we just looked at we were looking at a subset of three students being made from four available students. But while they seem to follow the same idea, we ended up with different answers!

Let's look a little more closely at the two examples we had and how the order of their items affected the answer:

- **Example 1:** In the race there is a first, second and third place. If the final outcome was R-T-V, this means that Ryan came first, Tim came second and Vince came third. R-V-T would NOT be the same because now Vince came second and Tim came in third. We can say that *ORDER MATTERS*.
- **Example 2:** If one of the athletes made the team, they're on the team. There is no placing or hierarchy on the team. This means that a team, R-V-T is the same as R-T-V, T-V-R, T-R-V, V-R-T and V-T-R. We can say that *ORDER DOESN'T MATTER*.

Knowing when order does and doesn't matter is an important part of working with permutations and combinations so let's get some practice before we learn how to use them!

Exercises I

1. For each of the following scenarios, state whether order matters or not:

   (a) The number of ways three distinct plants can be arranged on a window sill.

   (b) Selecting a combination on a combination lock.

   (c) A math student is given a list of 5 math problems and is asked to solve any 3 of the problems. How many different problem selections can the student make?

   (d) Mr E. Lipps has in front of him a circle, a square, and a rectangle. In how many ways can he select some of the shapes? (He may pick any number of the shapes)

   (e) How many permutations are there of the numbers 1 - 5 taken 2 at a time? List them.

   (f) How many different ways can the letters in STATISTICS be arranged?

Permutations

We use **Permutations** when *ORDER DOES MATTER*. We can solve questions using methods we learned from Counting I or we can use the following formulas depending on which of the cases the question falls under:

Basic Permutations (PB)

When we have n items and want to know how many ways we can rearrange these items:

$$n! = (n) \times (n-1) \times (n-2) \times ... \times (2) \times (1)$$

Notice that this is just the factorial formula

Example: Let there be the set of numbers $\{1, 2, 3, 4, 5, 6\}$.:

(a) How many permutations of the six numbers are there?

(b) How many permutations of the six numbers are there if the permutations must end in an even number?

Solution: Instead of writing out all the different permutations, let's use our knowledge of factorials:

(a) If we draw out how many spaces we have for digits, we see:

$$\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}$$

Filling in from left to right, there are 6 options for what we could place in the first digit. There are then 5 options for the second digit, AND then 4 options for the third digit...AND so on until there is one option for the last digit:

$$\boxed{\mathbf{6}}\boxed{\mathbf{5}}\boxed{\mathbf{4}}\boxed{\mathbf{3}}\boxed{\mathbf{2}}\boxed{\mathbf{1}}$$

There is is therefore $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720$ permutations of the set.

(b) If we draw out how many spaces we have for digits, and knowing that there are only 3 options for the last digit being even, $\{2, 4, 6\}$:

$$\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{\mathbf{3}}$$

With one even digit being taken, there are still 5 numbers that can fill the first digit, AND then 4 to fill the second...and so on until there is one left:

$$\boxed{\mathbf{5}}\boxed{\mathbf{4}}\boxed{\mathbf{3}}\boxed{\mathbf{2}}\boxed{\mathbf{1}}\boxed{\mathbf{3}}$$

There is is therefore $5 \times 4 \times 3 \times 2 \times 1 \times 3 = 360$ permutations of the set.

K-Permutations of N (PK)

When we have a set of n elements and we want to know how many ways we can arrange k of them:

$$_nP_k = \frac{n!}{(n-k)!}$$

Example: Ryan, Tim, Vince and Luc are in the finals for a track meet at the UW finals. List all the possible ways that the top 3 can be made:

Solution: Let's write out the entire set of combinations for our top 3:

$R\,L\,V$	$R\,V\,L$	$R\,L\,T$	$R\,T\,L$	$R\,T\,V$	$R\,V\,T$
$L\,R\,V$	$L\,V\,R$	$L\,V\,T$	$L\,T\,V$	$L\,T\,R$	$L\,R\,T$
$V\,R\,T$	$V\,T\,R$	$V\,L\,T$	$V\,T\,L$	$V\,L\,R$	$V\,R\,L$
$T\,R\,V$	$T\,V\,R$	$T\,L\,R$	$T\,R\,L$	$T\,L\,V$	$T\,V\,L$

There are therefore 24 different scenarios for the final outcome of the race.

In the example we looked at before, we had 4 runners total. We first had to find all the ways we could take 3 at a time and then find all permutations of these smaller sets of 3. Basically, we found the **3-Permutations of 4**.

What if we don't want to write out all the permutations?

Solution: Let's use the formula to solve for the total number of top 3 permutations:

$$\begin{aligned} _4P_3 &= \frac{4!}{(4-3)!} \\ &= \frac{4!}{1!} \\ &= \frac{4 \times 3 \times 2 \times 1}{1} \\ &= 24 \end{aligned}$$

There are therefore 24 different scenarios for the final outcome of the race.

This was a relatively easy task to write out if we wanted to because of the small amount of athletes in the race. Clearly writing out all the top 3 permutations for a New York Marathon with thousands of runners becomes much more difficult. Using factorial notation and division makes this much easier.

Permutations with Repetitions (PR)

When we have n items and want to know how many ways we can rearrange these items given that there are repeats, r, of an item or several items:

$$\text{One Repeat: } \frac{n!}{r!} \text{ or Multiple Repeats: } \frac{n!}{r_1!r_2! \text{ ... } r_n!}$$

Example: How many permutations exist using the letters of the following words:

(a) APPLIED

(b) STATISTICS

Solution: Let's investigate how many repetitions exist:

(a) Counting out the letters (and numbering the repeats) we have a total of 7 letters:

$$A \;\; P_1 \;\; P_2 \;\; L \;\; I \;\; E \;\; D$$

We see that there are 2 P's and that the following are the same for this and all permutations of the letters:

$$A \;\; P_1 \;\; P_2 \;\; L \;\; I \;\; E \;\; D \;\; = \;\; A \;\; P_2 \;\; P_1 \;\; L \;\; I \;\; E \;\; D$$

Since there are 2 P's we can write the formula as:

$$\frac{7!}{2!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 7 \times 6 \times 5 \times 4 \times 3 = 2520$$

There are therefore 2520 different permutations for the letters of APPLIED.

(b) Counting out the letters (and numbering the repeats) we have a total of 10 letters:

$$S_1 \;\; T_1 \;\; A \;\; T_2 \;\; I_1 \;\; S_2 \;\; T_3 \;\; I_2 \;\; C \;\; S_3$$

We see that there are 3 S's, 3 T's and 2 I's and if we use the formula:

$$\frac{10!}{3!3!2!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 3 \times 2 \times 1 \times 2 \times 1} = \frac{604800}{12} = 50400$$

There are therefore 50400 different permutations for the letters of STATISTICS.

Finding permutations with letters is not the only situation where we can use this technique. Whenever there are repeats of any items in the used set, we must use this technique.

Exercises II

1. State if the question requires Basic Permutations (PB), K-Permutations of N (PK) or Permutations with Repetitions (PR) to solve:

   (a) The number of ways three distinct plants can be arranged on a window sill.

   (b) Selecting a combination on a combination lock.

   (c) How many permutations are there of the numbers 1 - 5 taken 2 at a time. List them.

   (d) How many different ways can the letters in CALCULATOR be arranged?

   (e) How many ways can Jenny the Jeweller make a bracelet with 30 distinct beads?

Combinations

We use **Combinations** when *ORDER DOESN'T MATTER*. We can solve questions using methods we learned from Counting I or we can use the following formula:

When we have a set of items and want to know how many ways we can pick some of them, we write:

$$_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Example: At a cafeteria, a student is allowed to pick 4 items from the following list:

{ pop, juice, milk, water, burger, hotdog, vegetable soup, banana, orange, apple pie }

(a) How many ways can a student have a 4 piece meal?

(b) How many ways can a student have a 4 piece meal if they can only have one drink?

Solution: Let's investigate how many combinations exist:

(a) Using the formula, we have 10 items and want to choose 4:

$$_{10}C_4 = \binom{10}{4} = \frac{10!}{4!(10-4)!} = \frac{10!}{4!(6)!} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = \frac{5040}{24} = 210$$

There are 210 combinations that the student could have.

(b) This question follows the same idea except there are 4 drinks and he can only pick one, and then 6 other items from which he must pick the remaining 3:

$$\binom{4}{1} \times \binom{6}{3} = \frac{4}{1!3!} \times \frac{6}{3!3!} = 4 \times 20 = 80$$

There are 80 combinations where the student only has one drink.

Problem Set

1. Given the set { 1, 2, 3, 4 }:
   (a) How many permutations are there on the set? List them.
   (b) How many permutation on the set start with 2? List them.
   (c) How many permutations on the set have 1 and 2 beside each other? List them.

2. Evaluate the following:
   (a) $\binom{5}{3}$
   (b) $\binom{10}{6}$
   (c) $\dfrac{\binom{7}{6}}{\binom{4}{2}}$

3. A student club with 10 members wishes to select a president, a secretary and a treasurer from its membership. No member may be selected for more than 1 office. In how many ways can this be done?

4. Calculate the number of permutations on a set with 3 A's, 6 B's and 3 C's.

5. How many permutations of the numbers { 1, 2, 3, 4, 5, 6 }:
   (a) begin with an even number?
   (b) begin with an odd number and end with an even number?
   (c) begin with an even number and end with an even number?

6. A school has 380 female students and 120 male students. They must create a 5-person student council.
   (a) In how many ways can they do this?
   (b) In how many ways can they do this if it can only be made of girls?
   (c) In how many ways can they do this if there must be 4 boys and 1 girl?
   (d) In how many ways can they do this if there must be more girls than boys?

7. Brad the Bachelor has dwindled down his choices to 10 lucky bachelorettes. How many combinations are there:

   (a) If the show is getting cancelled and he needs to pick 4 finalists?

   (b) If the show says he must pick Beautiful Betty and 3 other finalists?

   (c) If he doesn't want to pick Smelly Sandra and must pick 4 finalists.

8. Given the set { A, A, A, B, B }:

   (a) Write out all the combinations.

   (b) Use the required formula to calculate the number of combinations and make sure that you were correct.

9. * Tim the Math Circles Assistant bought samosas, but didn't give Ryan any. Ryan's just going to steal them. How many ways can he steal some of Tim's samosas (**Hint:** He can steal any number of samosas):

   (a) If Tim has 3 samosas?

   (b) If Tim has 4 samosas?

   (c) If Tim has n samosas?

10. * In how many ways can 25 students be seated in a classroom with 25 desks? With 30 desks?

11. * * Izzy has returned from last week for more ice cream. He has k choices of flavours and can have pick n scoops. He does not need to have different flavours for his scoops, he can have multiples of one kind. How many combinations of ice cream can he have in each case? (Listing them may help)

    For example: If he had to choose between Chocolate and Vanilla and could take 2 scoops: he could take Vanilla-Vanilla, Chocolate-Vanilla or Chocolate-Chocolate, giving 3 total combinations.

    (a)
    - Ice Cream: Vanilla, Strawberry and Chocolate
    - Scoops Allowed: 2

    (b)
    - Ice Cream: Vanilla, Strawberry and Chocolate
    - Scoops Allowed: 3

    (c)
    - Ice Cream: Vanilla, Strawberry, Green Mint and Chocolate
    - Scoops Allowed: 3