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Grade 7/8 Math Circles

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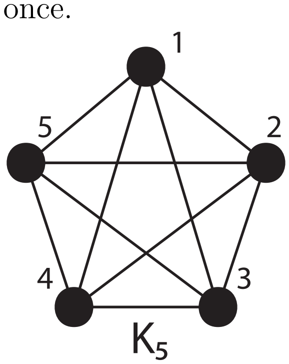
Graph Theory II- Solutions

"*" indicates challenge question

1. There are five teams in a tournament.

   (a) Use a graph to prove that there is a way for each team to play every other team once.

   i. How many games does each team play?
   Each team plays 4 games

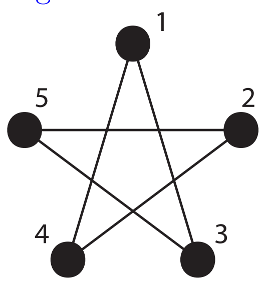
   ii. What is the total number of games played?
   There are 10 games played.

   (b) The organizers of the tournament would like to consider the different number of games each team can play (ie. how many games each team should play). Assuming each team must play at least one game, the options are for each team to play 1 game, 2 games, 3 games, or 4 games. Assume that each team has to play the same number of games, and that teams can't play each other more than once.

   i. Is it possible to draw a graph for all the different number of games? Explain. Be sure to use specific reasons you learnt in this lesson.
   1 and 3 are not possible. If there is an odd number of teams and each team plays an odd number of games, then the sum of the degrees will be an odd number. But the Handshake Lemma says that the number of edges is half of the sum of the degrees. Since the sum is odd, then the number of edges will not be a whole number. But we know this can not be true. For example, if each team played 1 game, the sum of the degrees will be 5. By the Handshake Lemma there should be 2.5 edges. However, we can not have half an edge.

ii. Draw one graph for each number of games that is possible.

The graph for 4 games is drawn in part (a). This is the graph for 2 games (note that it may be drawn in different ways, but each vertex must have two edges connected to it):

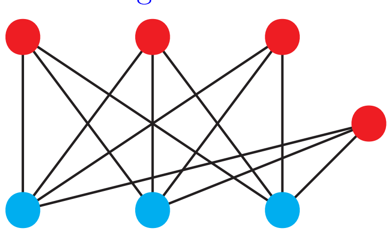


iii. Can you fix the scenarios from part i) by removing one of the assumptions? You must still assume that each team has to play at least one game. Answers may vary. Example: adding an extra game for one team.

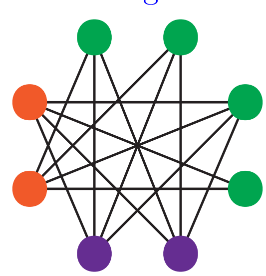
2. *How can you rearrange vertices to make it easier to see the minimum colouring. (Hint: think about grouping vertices according to what they are connected to) Answers may vary. You can group together vertices that are not adjacent, then make each group one colour

3. Find the minimum colouring for each of the following graphs. A good strategy for these questions is to colour one vertex, and assign a different colour to adjacent vertices by trying to use colours that have already been used and adding new colours when necessary.

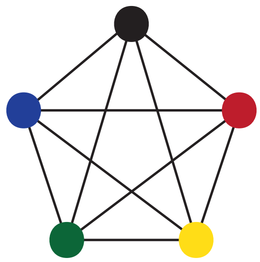
(a) 2-colouring as shown



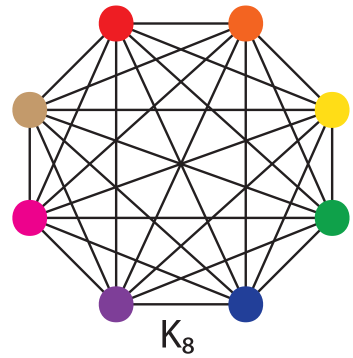
(b) 3-colouring as shown



(c) 5-colouring as shown



(d)

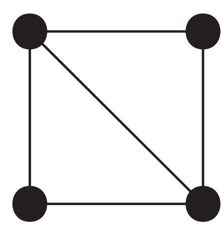


Remember that K_8 means that there are 8 vertices and each vertex is connected to the 7 other vertices. Since all the vertices are connected to each other, no two vertices can have the same colour. Therefore, it is an 8-colouring as shown.

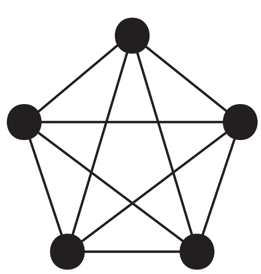
4. Draw a graph that is 1-colourable, 5-colourable, 10-colourable, 20-colourable, 50-colourable, 100-colourable, and 1000-colourable. Answers may vary. There can be more than 1 vertex, but since the graph must be 1-colourable none of the vertices can be connected. A sample answer is a single vertex.

5. Draw a graph that has a minimum colouring of (The easiest graph would be the complete graph (K_n) where n represents the minimum colouring needed)

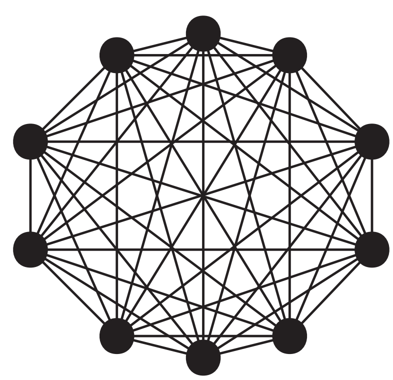
(a) 1 



(b) 3



(c) 5



(d) *10

6. Use the map of Europe to answer the following questions. When looking at the map, be careful not to mistakes rivers for boarders.

(a) Find the minimum colouring of the graph. (Hint: use the theorems) Since borders don't overlap, the map can be converted to a planar graph. Since the graph is planar, it can be coloured using four colours (according to the four-colour theorem).

(b) **Determine a colour for each country. Watch out for countries that that share a very small border. Don't forget about the small countries (you may need to research some countries if they are not clear on the map)Answers will vary

(c) Ryan wants to go to Sochi, Russia to see where the Olympics took place. He will fly to Portugal and drive to Russia. He doesn't want to waste too much time waiting at the border, so he would like to travel through the least number of countries possible. Use a BFST to find the best route (you may stop the BFST when you reach Russia). Answers may vary. Ryan should travel to 7 countries (including Portugal and Russia). A sample route is Portugal → Spain → France → Germany → Poland → Belarus → Russia

7. This is a map of where an airline flies. Use it to answer the following questions. Notice that this map already has a graph on it. However, you may wish to redraw the map as a graph to make it easier to see.

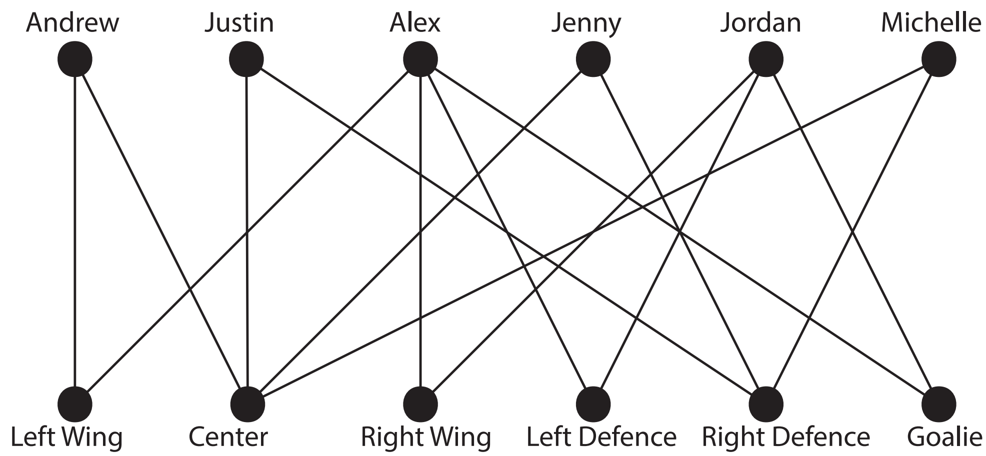
   (a) Find the minimum colouring. Colourings may vary, but should have exactly 3 colours

      i. How many "colours" did you use? The minimum colouring is 3

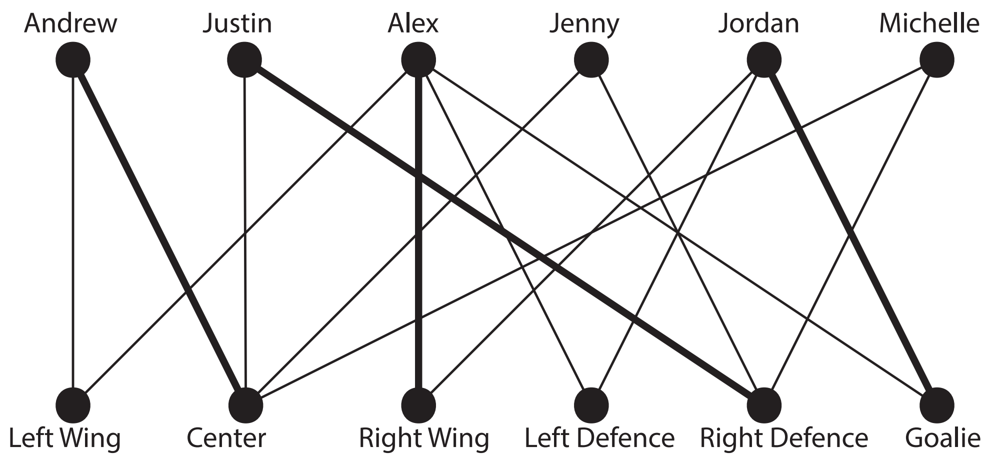
      ii. What does the colouring tell you about the flights? (Hint: think about direct flights) There are no direct flights between cities of the same colour. Note that this does not mean cities with a different colour have a direct flight.

   (b) Use a BFST to find the least number of flights needed to get from Calgary, Alberta to Halifax, Nova Scotia. How many flights do you need? 3 flights are needed.

8. **The Edmonton Eulers hockey team has 6 spots open (3 forward, 2 defence, and 1 goalie). Below is a graph of the players and the open spots. A player is adjacent to a spot if they can play that position. All the players have an equal skill level and the coach wants to fill as many spots as possible.



(a) Starting with the matching below, use the maximum matching algorithm to find the maximum matching. Matchings may vary.



(b) What is the maximum number of spots the coach can fill? The coach can fill 5 spots at most.

(c) Is there a different way the coach can fill the maximum number of spots? Yes, there are multiple ways to fill 5 positions

9. ***Prove the six-colour theorem. The following is a simplified version of an inductive proof. **Induction** is a proof method that starts with the simplest base case, and builds up to show it is true for any number. Think of induction as a prince trying to climb a ladder to get to his princess who is at the top of a castle. You want to prove that he can get onto the first step of the ladder. Then you can prove he can make it to the second, then the third, and so on. With mathematical induction, instead of proving that he can make it to a specific step we prove the first step and show that it works for a generalized step (usually represented with a variable). True induction is quite hard, so it is acceptable at this level to show the general idea.

   We start with 1 vertex. This is 6-colourable since we only need one colour. The same idea can be used for all planar graphs with 6 or less vertices.

   Next, we look at a graph with 7 vertices. From the theorem presented in the lesson, there must be a vertex with degree 5 or less. If we temporarily remove this vertex, we have a graph with 6 vertices. We have already proven that a graph with 6 vertices is 6-colourable, so we can colour this graph. Adding the 7th vertex back, we can use the colour of the vertex that it is not connected with to colour it. Since we have only used 6-colours, this graph with 7 vertices is 6-colourable.

   Using the same idea, we can start with 8 vertices and remove a vertex with degree 5 or less. Then we have a graph with 7 vertices. We have proven this is 6-colourable, so adding the 8th vertex back in, we can colour it by using one of the colour(s) that is not taken up by an adjacent vertex.

   Using this idea in general (this is the hard part), we have proven the base case, so we assume that the 6-colour theorem is true for $k-1$ vertices. Looking at a graph with k vertices, we can remove the vertex with degree 5 or less so we have $k-1$ vertices left. Since we assumed this is true, we can add the remaining vertex and colour it with the remaining colour.

10. ** Show that in a tournament with 25 people, at the end of the tournament, if x is the number of people who played an odd number of games, then x is even. Let's look at this situation as a graph with each team as a vertex and each edge as a game. If x is odd, then the sum of the degrees is odd since we are adding an odd number of numbers. According to the Handshake Lemma, dividing this number by two will give the number of edges or games. However, since this number is odd dividing it by two will not result in a whole number. Since the number of games (and edges) must be a whole number, this can not be true. Therefore, x can not be odd and must be even.

11. ** In the UW Information Game, 50 people are each given a unique piece of information. A person may call another person by one way-radio and transmit their information to the receiver, but it does not go both ways. For example, Mike can call Alice and give her his information (and thus she would have 2 pieces), but she cannot give him hers in the same call; she would have to make another call and then give it to him. However, Alice could then call Steve and give him all of her information (her original info, and the info she got from Mike). Then Steve would have 3 pieces of information. What is the minimum number of phone calls required so that everybody has all 50 pieces of information, and how do you achieve this? The minimum number of phone calls is 98 (49 to gather all the information together, and 49 to send the information to everyone). This can be achieved in a variety of ways. For example, all the information can be passed to one person who then sends it back to everyone

12. *** Show that in a house with 25 rooms, if every room has an odd number of doors then there must be an odd number of doors alongside the outside wall of the house. First let's convert the problem to a graph where each room is a vertex and each edge joins rooms that have a door between them (don't forget to add an extra vertex to represent the outside for a total of 26 vertices). Similar to question 10, if we add up the degrees of all the "inside" vertices (rooms) we will get an odd number since we are adding an odd number (of doors) an odd number (indoor rooms) of times. As we showed in question 10, this sum must be an even number. Therefore, the number of door the go outside must be an odd number so that the total sum of all the degrees is even.