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Grade 7/8 Math Circles

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Probability

Probability

Probability is the study of how likely an event is to occur. Several different definitions have been created to encompass different situations. The definitions are as follows:

Definitions of Probability

1. The **Classical Definition** states: *The probability of an event, A, occurring (where each outcome is equally likely) can be given by:*

$$\text{P(A)} = \frac{|\{\text{ways A can occur}\}|}{|\{\text{total outcomes}\}|}$$

Example: On a dice there a six face from 1 to 6. There is only one way to roll a 2 and 6 total outcomes. Therefore the probability of rolling a 2 is $\frac{1}{6}$.

2. The **Experimental Definition** states: *The probability of an event, B, occurring is based on data from a long series of repetitions of an experiment or process:*

$$\text{P(A)} = \frac{\text{number of trials in which the event occurred}}{\text{total number of trials in the experiment}}$$

Example: After rolling the above dice, out of your 36 trials, you rolled a 2 seven times. This is close to the expected value of six. You would say the experimental probability is $\frac{7}{36}$.

3. The **Subjective Definition** states: *The probability of an event is a measure of how sure the person making the statement is that the event will occur.*

Example: After hosting the weather for 20 years, Wally the Weatherman believes that there is 30 % chance of rain on July 30th every year. This is his opinion and based on his thoughts.

Sample Spaces and Events

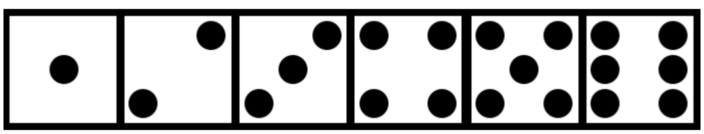
In this lesson, we will only focus on scenarios that follow the classical definition and therefore all examples will use outcomes which are equally likely due to using discrete sample spaces:

- **Sample Space**: the set of all possible outcomes of an experiment. We say a sample space is discrete if it contains a finite or a countably infinite number of outcomes. To be a **Probability Distribution** on a discrete sample space the following must hold true for each probability, p_i:

    1. $0 \leqslant p_i \leqslant 1$
    2. $p_1 + p_2 + ... + p_n = 1$

- **Event**: a subset of a sample space. Another way to think of this is that an event is a collection of outcomes that share some property.

Example: Let's look at a dice with the following faces:



We see that there are six possible outcomes that we could see if we were to roll the dice. In this case our sample space would be $\{ 1, 2, 3, 4, 5, 6 \}$. In the classical definition, this would be our set of total outcomes, and the **cardinality** or **size** of this set is the denominator.

A) What is the probability of rolling a 2?

The set of ways we can roll a 2 is { roll a 2 }. The size of this set is 1.

$$\text{P(rolling a 2)} = \frac{|\{\text{ways to roll a 2}\}|}{|\{\text{total outcomes}\}|} = \frac{1}{6}$$

B) What is the probability of rolling an even number?

The set of ways we can roll an even is { roll a 2, roll a 4, roll a 6 }. The size of this set is 3.

$$\text{P(rolling an even)} = \frac{|\{\text{ways to roll an even}\}|}{|\{\text{total outcomes}\}|} = \frac{3}{6} = \frac{1}{2}$$

Relationships

There are three main relationships between events that we will be working with. It is important to recognize the difference because the relationship of two events affects how they will be represented through operations.

- **Independent Events:** Two events such that occurrence of one does not affect the probability of the other.

  Ex: *Drawing a Jack from a standard deck of cards AND getting a head from a coin-toss*

- **Dependent Events:** Two events such that occurrence of one affects the probability of the other.

  Ex: *Flipping a coin and getting a head AND flipping a second coin and getting two heads over both*

- **Mutually-Exclusive Events:** Two events that cannot occur at the same time.

  Ex: *When flipping a coin it is impossible to get a head AND a tail on a single flip*

Exercises I

1. What is the sample space for rolling a fair six-sided die and flipping a fair coin?

2. State whether the following pairs of events are Mutually-Exclusive, Independent or Dependent (**Note:** assume all are without replacement):

   (a) Drawing a 7 from a deck of cards AND drawing a Jack next.

   (b) Drawing a 7 AND an 8 from a deck in one draw.

   (c) Getting a head on a coin-toss AND getting a tail on a different coin-toss.

3. Find the probability of the following events:

   (a) Drawing a Spade from a deck of cards

   (b) Drawing a face card (J, Q, K, A) from a deck of cards.

   (c) Drawing a 6 from a deck after three 6's have been removed from the deck.

   (d) Drawing a 6 from a deck after all the face cards have been removed.

Compound Probabilities

Product Rule

The **Product Rule** states:

> *If the probability of Action 1 occurring is p, and the probability of Action 2 occurring is q, then the probability of Action 1* ***AND*** *Action 2 occurring is p × q.*

We denote the probability of A **AND** B occurring as P(A ∩ B):

- **Example: Independent Events:** Find the probability of getting a head on a coin-toss and getting a tail on a different coin-toss.

  Solution: We know that when we have a coin, P(Head) = P(Tail) = $\frac{1}{2}$. We want to *Flip a Head* **AND** *Flip a Tail* so we see:

  $$\text{P(Head)} \times \text{P(Tail)} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

  Two events are independent if $\mathbf{P(A \cap B) = P(A)P(B)}$ like above.

- **Example: Dependent Events:** Find the probability of drawing a 7 from a deck of cards and then drawing a Jack on the next draw.

  Solution: Out of 52 cards there are four 7's. We can then say P(Draw7) = $\frac{4}{52}$ = $\frac{1}{13}$. Since a card has been drawn there are only 51 cards left, with four Jacks available. This means P(JackAfterDraw) = $\frac{4}{51}$. We want to *Draw a 7* **AND** *Draw a Jack After Removing 1 Card* so we see:

  $$\text{P(Draw7)} \times \text{P(JackAfterDraw)} = \frac{1}{13} \times \frac{4}{51} = \frac{4}{663}$$

  These events are not independent because P(Draw7) × P(DrawJack) $= \frac{1}{13} \times \frac{1}{13} = \frac{1}{169} \neq \frac{4}{663}$. The probability changes due to the dependence.

- **Example: Mutually-Exclusive Events:** Find the probability of drawing a 7 and an 8 from a deck in one draw.

  Solution: A card only has one value, and therefore they cannot be both a 7 and an 8. These are events that cannot happen at the same time. So without mathematical derivation, we know that the probability of this ever happening is 0. **In fact, for all Mutually-Exclusive Events, the probability of both occurring is 0. We can say $\mathbf{P(A \cap B) = 0}$**

Sum Rule

The **Sum Rule** states:

> *If the probability of Action 1 occurring is p, and the probability of Action 2 occurring is q, then the probability of Action 1* ***OR*** *Action 2 occurring is p + q - (p × q).*

We denote the probability of A **OR** B occurring as P(A ∪ B):

- **Example: Independent Events:** Find the probability of drawing a 5 or a spade on a single draw.

  Solution: If we use the formula P(A ∪ B) = P(A) + P(B) - P(A ∩ B), we know that since these are independent events, we must change it to P(A ∪ B) = P(A) + P(B) - P(A)P(B). We can see that P(Draw5) = $\frac{4}{52} = \frac{1}{13}$ and P(DrawSpade) = $\frac{13}{52} = \frac{1}{4}$. However, the 5 of Spades gets double counted, so we must find the probability of *Draw5* ***AND*** *DrawSpade*. Now P(A ∩ B) = $\frac{1}{13} \times \frac{1}{4} = \frac{1}{52}$.

  $$\text{P(Draw5 or Draw Spade)} = \frac{1}{13} + \frac{1}{4} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

- **Example: Mutually-Exclusive Events (1):** Find the probability of getting a head or a tail on a single coin-toss.

  Solution: If we use the formula P(A ∪ B) = P(A) + P(B) - P(A ∩ B), we know that since these are mutually-exclusive events, we must change it to P(A ∪ B) = P(A) + P(B). We also know that P(Head) = P(Tail) = $\frac{1}{2}$. Substituting we get:

  $$\text{P(A} \cup \text{B)} = \text{P(Head)} + \text{P(Tail)} = \frac{1}{2} + \frac{1}{2} = 1$$

  2 mutually-exclusive events are called Jointly Exhaustive if their sum is 1.

- **Example: Mutually-Exclusive Events (2):** Find the probability of getting a 1 or a 2 on a single dice roll.

  Solution: If we use the formula P(A ∪ B) = P(A) + P(B) - P(A ∩ B), we know that since these are mutually-exclusive events, we must change it to P(A ∪ B) = P(A) + P(B). We also know that P(Roll1) = P(Roll2) = $\frac{1}{6}$. Substituting we get:

  $$\text{P(A} \cup \text{B)} = \text{P(Roll1)} + \text{P(Roll2)} = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

Complements (Part 1)

Basic Complements

Knowing the probability of something occurring allows us to also find the probability of it's complement, or in other words: the probability of it **NOT** occurring.

Let's look at an example and some interesting things to note about an event and its complement:

Example: Given a fair dice, let A be the event of rolling a 2. We know the probability of this event occurring is $\frac{1}{6}$. If we want to find the complement of A, denoted as $\overline{A}$ - which is the event of **NOT** rolling a 2, we can look at it as rolling a 1 **OR** 3 **OR** 4 **OR** 5 **OR** 6. Mathematically we see:

$$P(\overline{A}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{5}{6}$$

While the complement can be the sum of all events that are **NOT** A, let's look at our knowledge of mutually exclusive events.

"Rolling a 2" and *"Not rolling a 2"* can never occur at the same time. We can think of A and $\overline{A}$ as mutually-exclusive events. Since rolling a 2 and not rolling a 2 (rolling a 1, 3, 4, 5 or 6) makes up all of the events of rolling a dice, we can also think of the two events as jointly exhaustive events such that: $P(A) + P(\overline{A}) = 1$

Rearranging this statement, we get that the formula for finding the complement of some event, A, is:

$$P(\overline{A}) = 1 - P(A)$$

Looking at our question, we confirm that our formula is correct:

$$P(\overline{A}) = 1 - P(A) = 1 - \frac{1}{6} = \frac{5}{6}$$

While finding the complement in this case was pretty straightforward and was sued in a situation where we could've just found the sum of all events that weren't A, there is a more useful application known as the indirect method.

Binomial Distribution

In experiments where there are independent successes and failures repeated numerous times, we can find probabilities of a certain amount of successes out of a certain number of trials using the **Binomial Distribution**. The formula for the Binomial Distribution is as follows:

$$P(\text{K Successes out of N Trials}) = \binom{n}{k}(S)^n(F)^{n-k} = \frac{n!}{k!(n-k)!}(S)^n(F)^{n-k}$$

The meaning of each of the parts:

- n = total number of trials
- k = total number of successes
- S = probability of a success, also called (p)
- F = probability of a failure, also called $(1-p)$

Example: The probability that it will rain on a certain day is 30%. What is the probability that it will rain on 3 days out of a 5-day work week?

Solution: Let's use the formula by deciding what all of our parts should be if we consider a rainy day as a success:

- n = total trials = 5 days
- k = total successes = 3 days
- n - k = total failures = 2 days
- S = probability of raining = 30% or 0.3
- F = probability of not raining = 70% or 0.7

$$P(\text{3/5 Days of Rain}) = \binom{5}{3}(0.3)^3(0.7)^2 = 0.1323 \text{ or } 13.23\%$$

So why exactly does this formula work? If we think of the question as "*What is the probability that we will have a RainyDay* ***AND*** *a RainyDay* ***AND*** *a RainyDay* ***AND*** *a NonRainyDay* ***AND*** *a NonRainyDay*" we could use the Product Rule to give us:

$$(0.3) \times (0.3) \times (0.3) \times (0.7) \times (0.7) = (0.3)^3 \times (0.7)^2$$

However, we have to take into account the number of different ways we could pick 3 out of 5 days. As seen if we write it out, with R being a RainyDay and N being a NotRainyDay, there is 10. Here we can use the Choose Formula:

$NNRRR$	$NRNRR$	$NRRNR$	$NRRRN$	$RRRNN$
$RNNRR$	$RNRNR$	$RNRRN$	$RRNNR$	$RRNRN$

Complements (Part 2)

Indirect Method

Sometimes finding the complement of the event we are looking for is in fact easier and can help us find the probability of the event in question. Rearranging the complement formula we get:

$$P(A) = 1 - P(\overline{A})$$

Example: The probability that it will rain on a certain day is 30%. What is the probability that it will rain **at least one day** out of a 5-day work week?

Solution: Let's look at each separate probability using the binomial distribution formula:

- $P(0/5 \text{ Days of Rain}) = 16.81\%$
- $P(1/5 \text{ Days of Rain}) = 36.01\%$
- $P(2/5 \text{ Days of Rain}) = 30.87\%$
- $P(3/5 \text{ Days of Rain}) = 13.23\%$
- $P(4/5 \text{ Days of Rain}) = 2.83\%$
- $P(5/5 \text{ Days of Rain}) = 0.24\%$

These all add up to 100% and can be thought of as the samples space of the 5 days of rain. We can write:

$$\mathrm{P}(0) + \mathrm{P}(1) + \mathrm{P}(2) + \mathrm{P}(3) + \mathrm{P}(4) + \mathrm{P}(5) = 1$$

Finding the probability of **at least one day** means 1 day or more. This means we want the probability of 1 day **OR** 2 days **OR** 3 days **OR** 4 days **OR** 5 days.

Since we have all the probabilities given to us we could do:

$$\mathrm{P}(1) + \mathrm{P}(2) + \mathrm{P}(3) + \mathrm{P}(4) + \mathrm{P}(5) = 0.8319 \text{ or } 83.19\%$$

But, we know the complement of this set is just P(0), so if we we rearrange the formula from before:

$$\begin{aligned} P(0) + P(1) + P(2) + P(3) + P(4) + P(5) &= 1 \\ P(1) + P(2) + P(3) + P(4) + P(5) &= 1 - P(0) \\ 0.8319 &= 1 - P(0) \end{aligned}$$

This is called the **indirect method** because instead of adding up everything we want, we subtract everything we don't want. This becomes extremely useful for larger sets of probabilities, for example if we had 20 days and wanted to find the probability of at least 2 being rainy.

Problem Set

1. Which definition of probability is being used here?
   (a) Participating in a raffle.
   (b) Being 90% sure you passed your math test.
   (c) Testing several products in a production line to see if they are defective.

2. Find the missing probability that makes the set a Probability Distribution:
   (a) $P(0) = \frac{1}{7}$, $P(1) = \frac{2}{7}$, $P(2) = X$, $P(3) = \frac{3}{7}$
   (b) $P(0) = X$, $P(1) = \frac{2}{5}$, $P(2) = \frac{3}{7}$

3. $P(A) = 0.6$ and $P(B) = 0.4$. If $P(A \cap B) = 0.25$ are:
   (a) A and B be independent?
   (b) A and B jointly exhaustive?

4. When rolling two fair dice, what is the probability that the sum is:
   (a) greater than 5?
   (b) less than 2?
   (c) equal to 8?
   (d) less than or equal to 12?

5. A weighted coin is altered so the probability of it landing on a head for each flip is $\frac{5}{7}$. The trick coin is flipped 3 times. What is the probability of getting a tail on the first flip and heads on the next two flips?

6. A regular coin is flipped and then a card is randomly drawn from a standard deck of 52 cards.
   (a) Determine the probability of flipping a head, then drawing a diamond.
   (b) Determine the probability of flipping a head, then drawing a diamond or a heart.

7. In a coin toss, what is the probability that heads is flipped exactly two times out of three tosses? What if the coin is weighted so that the probability that heads occurs is 0.6?

8. What is the probability of winning the Lotto 6/49? (**Note:** Lotto 6/49 is played such that there is 49 numbers and you must pick 6 without repeats.)

9. Mark has a bag that contains 3 black marbles, 6 gold marbles, 2 purple marbles, and 6 red marbles. Mark adds a number of white marbles to the bag and tells Susan that if she now draws a marble at random from the bag, the probability of it being black or gold is $\frac{3}{7}$. How many white marbles does Mark add to the bag?

10. * In a deck of 52 cards, how many ways can you choose 5 cards having at least 2 kings?

11. * In Canada, the probability that someone plays baseball and/or hockey is 0.79. The probability that someone plays just hockey is 0.6 and the probability that some plays baseball and hockey is 0.15. What is the probability that some plays only baseball?

12. * A credit card PIN of length 4 is formed by randomly selecting (with replacement) 4 digits from the set 0 - 9. Find the probability:

    (a) the PIN is even

    (b) the PIN has only even digits

    (c) ** the PIN contains at least one 1

13. ** A student randomly guesses the correct answer in a multiple choice quiz of 10 question. If each question has 5 choices, what is the probability the student gets exactly 7 correct? What is the probability the students gets more than 7 correct?

14. * The 10,000 tickets for a lottery are numbered 0000 to 9999. A four-digit winning number drawn is and a prize is paid on each ticket whose four-digit number is any arrangement of the number drawn. For example, if the winning number 0011 is drawn, the grand prize is split between the people who hold 0011, 0101, 0110, 1001, 1010 and 1100. What is the probability of winning with:

    (a) 6446?

    (b) 7843?

15. ** A company owns 300 PS4 consoles. 8% of the consoles do not work. You randomly select 25 consoles.

    (a) What is the probability that all consoles are working?

    (b) What is the probability that 1 or 2 consoles are not working?

    (c) What is the probability that more than 3 consoles are not working?