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Grade 7/8 Math Circles

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Sets

Sets

A **set** is a collection of distinct objects. All the elements that belong to a set are called **members** of that set. Elements can be numbers, letters, symbols, essentially anything that would belong to your desired set.

Representing Sets

To denote a set we use *braces* or *curly brackets*, to enclose a list of elements. However, there are two primary ways that we can represent a set. Let's look at these with a set containing the numbers: 2, 4, 6, 8, 10.

- Represented as a list of objects:

$$\{2, 4, 6, 8, 10\}$$

- Represented as a description:

$$\{\text{Even Numbers Between 1 and 11}\}$$

While describing the elements contained in a set seems useless when the set is as small as the one above, it does become useful when describing sets that would be more difficult to write out completely.

For example if you wanted to write out the set of all positive odd numbers, you'd hopefully realize that you have an infinite amount of numbers to work with. Having a pattern or rule allows us to effectively represent large sets. Representing all positive odd numbers can be simplified as shown below:

$$\{\text{Odd Numbers}\} \text{ or } \{1, 3, 5, 7, 9...\}$$

Throughout this lesson we will be constantly assigning sets an arbitrary letter for ease of reading. If we were to say that A *is the set of all positive odd numbers*, we can now state:

$$A = \{\text{Positive Odd Numbers}\}$$

Size of a Set

The **size** or **cardinality** of a set is the number of elements contained in the set. The notation for the size of a set is to place vertical bars around the set. Let's look at the three cases we can have for a set size:

- A set with a finite number of elements:
  Let A be the set $\{2, 4, 6, 8, 10\}$, we can then write $|A| = 5$, since A has 5 elements.

- A set with no elements:
  Let B be the set {People Who Have Been to Neptune}. Since it is safe to assume that up to today nobody has visited Neptune, we know that this set has no elements. We call this the *empty set* or *null set*, and if we were to try and write it out it would look like this: { } , but we can also represent it with a $\emptyset$.

- A set with infinite elements:
  Let C be the set {Odd Numbers}. We've already discussed that this set goes on forever, so we represent its size with the symbol for infinity, ∞.

Elements of a Set

When we want to show that something is a member of a set, we use the $\in$ symbol. On the other hand if something is not part of a set, we use $\notin$. For example Ryan N $\in$ {Math Circle Assistants} , but Ryan N $\notin$ {Girls} . Two important rules we need to remember with sets are **uniqueness** and **equality**:

- **Uniqueness** means that all elements only appear once in the set. This is where being descriptive becomes useful for if Timmy T, Billy B and Timmy R are playing on a playground and we wanted to make a set, D, of kids playing on the playground, we could make a set using only first names: {Timmy, Billy}. But this doesn't work because we know that there are 3 kids on the playground but our set leads people to believe there are only 2. So by including last names, and letting $D =$ {Timmy T, Timmy R, Billy B}, we see $|D| = 3$ as we knew it should.

- **Equality** means that two set's elements are exactly the same regardless of order. This means that the sets {2, 4, 6, 8} and {8, 6, 4, 2} are equal because they have the exact same elements.

Universal Set, Subsets and Supersets

Lets look at four sets Q, R, S and T where:

$$Q = \{\text{Felines}\}$$
$$R = \{\text{Jaguar, Cougar, Leopard, Lion, Panther}\}$$
$$S = \{\text{Leopard, Panther}\}$$
$$T = \{\text{Cat, Lion}\}$$

We know that none of these sets are equal but we can see that there are some sets whose entire set of elements are in another. Let's look at some of the relationships we can find:

- If we look at S, we can see that all of its elements are also in R, but R has elements that are not in S. We can then say that S is a **subset** of R, and R is a **superset** of S. We show this relationship mathematically as $S \subseteq R$.
- We can also hopefully see that since Q is the set of all feline creatures, it contains both R, S and T within it so that both $R \subseteq Q$, $S \subseteq Q$ and $T \subseteq Q$ are true statements!

This is in fact a special set. If we were focusing on all feline creatures, the set Q, since it is the set of all feline creatures, becomes our **Universal Set** and can be represented by $\mathbb{U}$.

Exercises I

1. Are the following statements true or false? If false, explain why:
   (a) Is $\{1, 1, 2, 3, 4, 5\}$ a valid set based on what we learned in this lesson?
   (b) Is the set $\{1, 3, 5, 7, 9\}$ the same as the set $\{1, 3, 5, 9, 7\}$?
   (c) Is the set $\{3, 6, 9, 12, 15\}$ the same as the set $\{3, 6, 9, 12, 15...\}$?
2. Fill in the blanks with a $\in, \ni, \subseteq, \supseteq$ or $=$ to make the sentence true:
   (a) Dark Red __ { Shades of Red }
   (b) { Shades of Blue } __ { Colours } __ Green
   (c) Dog __ { Pets } __ Cat
   (d) { 100, 99, 98 ... 2, 1 } __ { 1, 2, ... 98, 99, 100 } __ { 2, 4, ... 98, 100 }
   (e) { Multiples of 2 } __ { ...-2, 0, 2, 4, 6, 8 ... } __ { Positive Even Numbers }

Operations with Sets

Just like many different mathematical concepts, sets have their own unique operations which we will primarily investigate through the use of Venn Diagrams.

Union

The **union** of two sets A and B is usually thought of as "*What elements are in A **or** B?*". While in the English language "or" would be equivalent to "in either A or B", mathematically it means we are looking for elements that are in **A or B or in both**. When writing out mathematical sentences we can ask the same question using "$\cup$": $A \cup B$

When finding the unions of any number of sets, it's important to remember our uniqueness:

$$\text{Let A} = \{\text{Red, Orange, Yellow}\}$$
$$\text{Let B} = \{\text{Red, Blue, Purple}\}$$

If we make $C = A \cup B$, you might think we have to write: $C = \{\text{Red, Orange, Yellow, Red, Blue, Purple}\}$. But we learned that elements can only be in the set once, so we can think of Union as writing all the unique items in A and B: $C = \{\text{Red, Orange, Yellow, Blue, Purple}\}$.

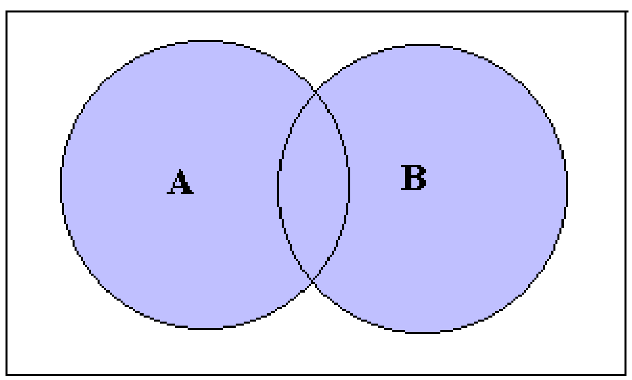


Figure 1: The Union of Two Sets

Intersection

The **intersection** of two sets A and B is usually thought of as "*What elements are in A **and** B?*". This means we are looking for elements that are in **both A and B**. When writing out mathematical sentences we can ask the same question using "$\cap$": $A \cap B$

While with unions we simply had to look for all unique elements in each set, with intersections we must write down only what is in both, which in some cases, may be nothing:

$$A = \{\text{Red, Orange, Yellow}\}$$
$$B = \{\text{Red, Blue, Purple}\}$$
$$C = \{\text{Green, Orange, Yellow}\}$$

If we now write out $A \cap B$, $A \cap C$ and $B \cap C$, we see that the intersection of two sets can even be the empty set!:

$$A \cap B = \{\text{Red}\}$$
$$A \cap C = \{\text{Orange, Yellow}\}$$
$$B \cap C = \{\}$$

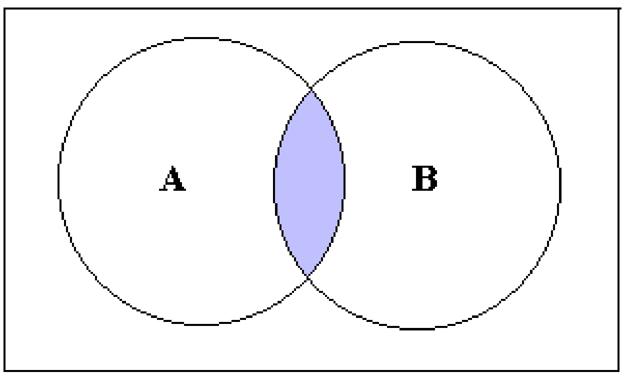


Figure 1: The Intersection of Two Sets

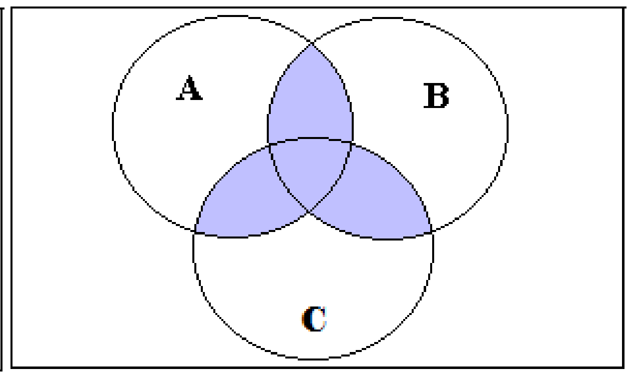


Figure 2: The Intersection(s) of Three Sets

Complements

In any question we approach, once we are given our Universal Set and its subsets, we can define the **complements** of any of the known subsets. The complement of A, a subset of $\mathbb{U}$, is the set of all elements that are **not** in A but are in $\mathbb{U}$. We show this mathematically by placing a bar over the set. Let's look at an example:

$$\mathbb{U} = \{\text{Red, Yellow, Blue, Green, Purple, Orange}\}$$
$$B = \{\text{Red, Blue, Yellow}\}$$
$$\overline{B} = \{\text{Green, Purple, Orange}\}$$

The Universal Set becomes important with complements for limiting sets. For example, if we have the set { Red, Blue }, and we don't limit our Universal Set, the complement can literally be anything, but if we limit our Universal Set to $\mathbb{U} = \{\text{Colours}\}$, it allows us to focus on what our complements can be for our specific scenario.

Exercises II

1. Let A = { 1, 3, 5, 7, 9 }, B = { 2, 4, 6, 8, 10 } and C = { 2, 3, 5, 7, 11 }. Write out the following sets if $\mathbb{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$:

   (a) $A \cup C$

   (b) $B \cap C$

   (c) $(A \cap C) \cup B$

   (d) $\overline{A}$

2. Let $R = \{$ Numbers Between 2 and 5, and Numbers Greater than 9$\}$. Find $\overline{R}$.

Complements of Unions and Intersections

We have been working with simple sets and their unions or intersections so far. Sometimes it is useful or necessary to find things that are not in sets that we made from unions or intersections.

To look at these sets, we must use **De Morgan's Laws**. His laws become really important when finding the complements of unions and intersections because it changes the operations we use:

$$\overline{\cap} \text{ becomes } \cup$$
$$\overline{\cup} \text{ becomes } \cap$$

Let's go through an example step-by-step to understand where this is useful:

Find the complement of $A \cap B$:

Let's first put a bar over the entire statement to give us: $\overline{A \cap B}$

If we split up the bar we see: $\overline{A} \,\overline{\cap}\, \overline{B}$

From above we know that $\overline{\cap}$ becomes $\cup$ so we get: $\overline{A} \cup \overline{B}$

We now know that the complement of $A \cap B$ is the union of $\overline{A}$ and $\overline{B}$!

Inclusion-Exclusion Principle

While counting the number of elements for a single set or even an intersection of sets is a relatively easy task, counting the number of elements for the union of two or more sets leads us to a common problem: double counting.

Let's say there are 30 students in a class. When the teacher asks their students if they like Coke or Pepsi, she gets the following results (assuming everyone likes at least one): 18 like Coke, 14 like Pepsi and 2 like both. When she counts up all her tallies, she gets 34, but she only has 30 students in her class. How is this possible?

We know that a union of two sets, A and B, contains the elements who belong to A, B and both A and B. If we were to count each set separately we would be double counting the elements that were in both A and B. The formula we use is:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Let's use this for our example:

We'll let $A = \{\text{Students Who Like Coke}\}, B = \{\text{Students Who Like Pepsi}\}$ and $A \cap B = \{\text{Students Who Like Both}\}$
Using our formula we get the expression:

$$|A \cup B| = 18 + 14 - 2$$
$$|A \cup B| = 30$$

This concept works for larger sets of unions as well:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Why Are Sets Important To Us?

The next two weeks we will begin to look at counting rules commonly used in Probability. The classical definition of the probability of some event, A, occurring where every outcome is equally likely is:

$$P(A) = \frac{|\{\text{ Ways event A can occur }\}|}{|\{\text{ All outcomes }\}|}$$

The probability of the event occurring is made up by the division of two sets. By knowing common set operations, we'll be able to learn these counting rules effectively and even use some of the techniques we have learned today!

Problem Set

1. Give 5 elements of the following sets, or explain why you can't. Are there any pairs of subsets and supersets?:
   (a) {Multiples of 3} $\cap$ {Multiples of 4}
   (b) {Integers greater than 4} $\cap$ {Integers less than 6}
   (c) {Factors of 100} $\cap$ {Factors of 20}
   (d) {Even Numbers} $\cap$ {Prime Numbers}
   (e) {Letters in the word "Because"}$\cap$
   {Letters in the poem "The Quick Brown Fox Jumps Over the Lazy Dog}
2. Let $\mathbb{U}$ be the Universal Set and A be a set within the Universal Set, in words:
   (a) What is $\overline{\mathbb{U}}$?
   (b) What is $A \cap \overline{A}$?
   (c) What is $A \cup \overline{A}$?
   (d) What is $\overline{\overline{A}}$?
   (e) What is $\{\ \} \cap \{\ \}$?
3. Like we did in the *Complements of Unions and Intersections* section, find the complement of $A \cup B$ step-by-step:
4. (a) Draw a Venn Diagram where A intersects B:
   (b) Shade in $\overline{A \cup B}$.
   (c) Shade in $\overline{A \cap B}$.
   (d) Shade in $\overline{A}$.
   (e) Shade in $A \cup \overline{B}$.
5. (a) Draw a Venn Diagram where A is a subset of B:
   (b) Shade in $A \cap B$.
   (c) Shade in $A \cup B$.
   (d) Shade in $\overline{A \cap B}$.
   (e) Shade in $A \cap \overline{B}$.

6. In a group of 60 people, 27 like cold drinks and 42 like hot drinks and each person likes at least one of the two drinks. How many like both hot drinks AND cold drinks?

7. Each student in a class of 40 plays at least one card game: Euchre, GoFish and Crazy8s. 18 play Euchre, 20 play GoFish and 27 play Crazy8s. 7 play Euchre and GoFish, 12 play GoFish and Crazy8s and 4 play Euchre, Crazy8s and GoFish.

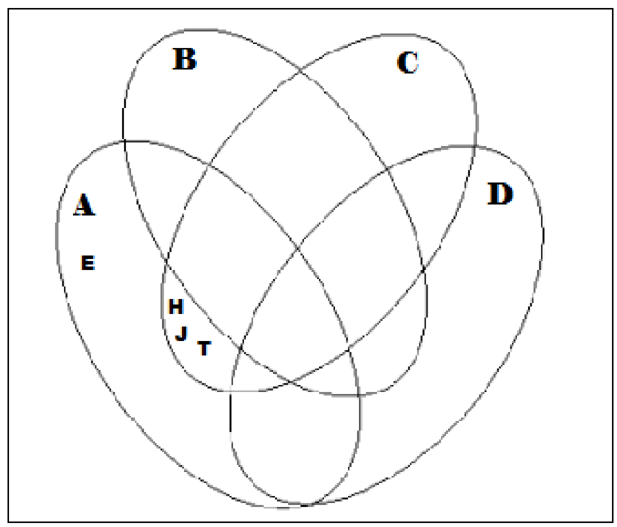
   (a) Find the number of students who play Euchre and Crazy8s. (Hint: Use the Inclusion-Exclusion Principle for three sets)

   (b) How many students play only Euchre?

8. * Is the size of the set {Even Numbers} equal to the size of the set {Odd Numbers}?

9. Let $A = \{E, H, J, T, S, G, Z, N\}$, $B = \{L, Y, U, I, O, Z, N, R, K\}$, $C = \{G, Q, N, R, K, Y, F, Z, S, H, J, T\}$ and $D = \{U, I, N, G, F, X, R, Z, K, V\}$:

   (a) Redraw the 4-set Venn Diagram shown to the right, and place all the elements in their proper section.



   (b) How many of the sections in the diagram to the right represent the intersection of 2 sets? 3 sets? All 4 sets?

   (c) What elements are in the set $(A \cap B) \cup (C \cap D)$? Shade this in on a Venn Diagram.

   (d) What elements are in the set $(\overline{B} \cap \overline{C}) \cap (A \cup D)$? Shade this in on a Venn Diagram.

10. In a restaurant, 40 people order fries. 13 people put salt on their fries, 28 people put ketchup on their fries, and 6 people do both. How many people eat their fries plain?

11. * Write the Inclusion-Exclusion Principle for $|A \cup B \cup C \cup D|$.

12. * Construct a Venn Diagram: 200 volleyball players were asked which of these moves they considered their weakest move(s): the serve, the bump, the spike.

   - 20 players said none of these were their weakest moves.
   - 30 players said all three of these were their weakest moves.

- 40 players said their serve and spike were their weakest moves.
- 40 players said that only their serve and bump were their weakest moves.
- 15 players said that their bump but not their spike was their weakest move.
- 52 players said that only their spike was their weakest move.
- 115 players said their serve was their weakest move.

13. * * Construct a Venn Diagram: 150 people at a Justin Bieber concert were asked if they knew how to play piano, drums or guitar.

    - 18 people could play none of these instruments.
    - 10 people could play all three of these instruments.
    - 77 people could play drums or guitar but could not play piano.
    - 73 people could play guitar.
    - 49 people could play at least two of these instruments.
    - 13 people could play piano and guitar but could not play drums.
    - 21 people could play piano and drums.

14. * * Is the size of the set $\{$Numbers from 0 to 1$\}$ equal to the size of the set $\{-\infty$ to $\infty\}$? (Hint: Remember that there are decimal numbers between 0 and 1)

15. * * Write the Inclusion=Exclusion Principle for $|A \cup B \cup C \cup D \cup E|$.