Intermediate Math Circles
Wednesday November 05 2014
Equations and Inequalities with One Variable

Inequalities:

The statement $a < b$ means:

- The number a is strictly smaller than the number b.
- On the number line, a must be to the left of b.
- $b = a + r$, where $r > 0$.

The statement $a \leq b$ means:

- The number a is smaller than or equal to b.
- On the number line, a must be to the left or on b.
- $b = a + r$, where $r \geq 0$.

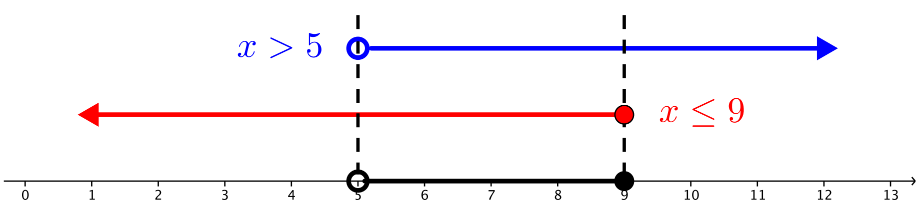
Compound Inequalities:

Example 1: Given $x > 5$ and $x \leq 9$, show the solution on a number line and then algebraically, using set notation.

When two or more inequalities are combined, the inequality is referred to as a **compound inequality**.

If the word **and** is used, then the solution must satisfy both inequalities. If the word **or** is used the solution satisfies either or both inequalities.

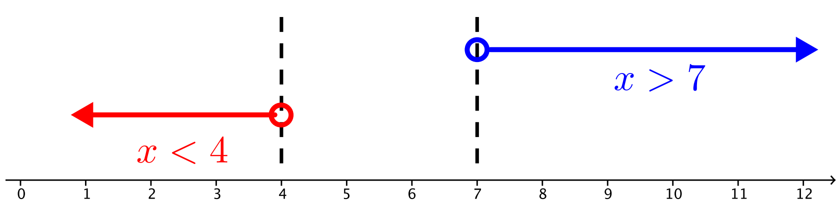
We will first represent the solution on a number line.



$$\{x \mid 5 < x \leq 9,\ x \in \mathbb{R}\}$$

We want the set of values for x that make both inequalities true at the same time. In fact we are looking for the intersection of the two inequalities.

Example 2: Given $x > 7$ and $x < 4$, show the solution on a number line and then algebraically, using set notation.



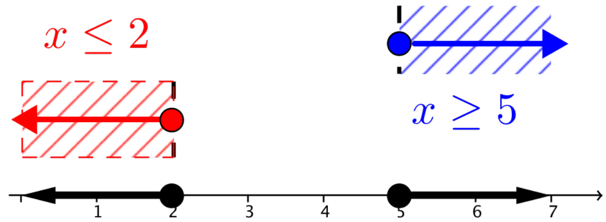
$$\text{No solution, } x \in \{\ \}$$

Drawing the number line, it is clear that there are no values of x which satisfy both inequalities at the same time. Therefore, there is no solution or the solution is the null set.

We also have compound inequalities which use “or” instead of “and”.

Example 3: Given $y \geq 5$ or $y \leq 2$, show the solution on a number line and then algebraically using set notation.

We need at least one of the inequalities to be satisfied, not necessarily both at the same time.

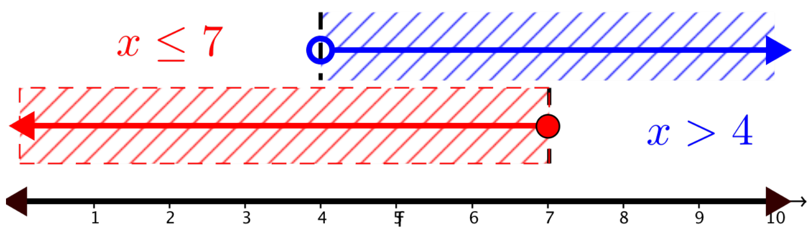


$$\{x \mid x \leq 2 \cup x \geq 5,\ x \in \mathbb{R}\}$$

We see that as long as y is not between -2 and 5, this compound inequality is satisfied.

The symbol $\cup$ is used instead of the word or, and stands for the word union.

Example 4: Given $x \leq 7$ or $x > 4$, represent the solution using a number line. Write the solution in set notation.



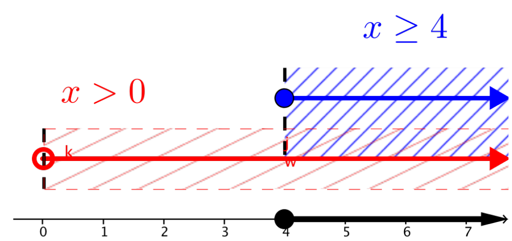
$$\{x \mid x \in \mathbb{R}\}$$

Example 5: Represent the solution to each of the following on a number line and in set notation.

a) $x > 0$ and $x \geq 4$

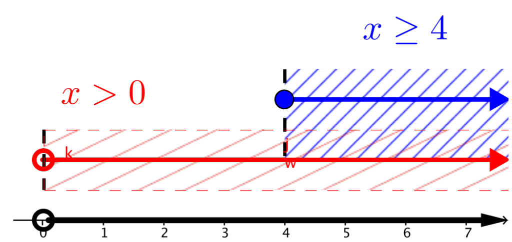
b) $x > 0$ or $x \geq 4$

a) We are looking for a solution that satisfies BOTH $x > 0$ and $x \geq 4$. Any number 0 or less does not satisfy either inequality. Any number above 0 but below 4 only satisfies $x > 0$ but not $x \geq 4$. Only numbers greater than or equal to 4 satisfy both inequalities. Therefore, the solution is $\{x \mid x \geq 4, x \in \mathbb{R}\}$.



$$\{x \mid x \geq 4,\ x \in \mathbb{R}\}$$

b) We are looking for a solution that satisfies either $x > 0$ or $x \geq 4$ or both. Any number 0 or less does not satisfy either inequality. Any number above 0 satisfies $x > 0$ and is in the solution. Numbers greater than or equal to 4 satisfy both inequalities. Therefore, the solution is $\{x \mid x > 0, x \in \mathbb{R}\}$.



$$\{x \mid x > 0,\ x \in \mathbb{R}\}$$

Some Rules for Working with Inequalities:

1. Adding or subtracting any number from both sides of an inequality preserves the inequality.

   Example. Suppose that $a = 4$ and $b = 19$. We see that $a < b$. If $c = 16$, show that $a + c < b + c$.

   $$4 < 19$$
   $$4 + 16 = 20 \quad \text{and} \quad 19 + 16 = 35$$
   $$20 < 35 \qquad \text{True}$$

   Example. Suppose that $a = 4$ and $b = 19$. We see that $a < b$. If $c = 5$, show that $a - c < b - c$.

   $$4 < 19$$
   $$4 - 5 = -1 \quad \text{and} \quad 19 - 5 = 14$$
   $$-1 < 14 \qquad \text{True}$$

   Prove: If $a < b$, then, for some $c \in R$, $a + c < b + c$.

   Proof. Since $a < b$ there exists some $r > 0$ such that

   $$b = a + r, \text{ where } r > 0$$

   For equalities we can add any number to both sides and still have the equality be true. If we add c to both sides we get:

   $$b + c = (a + c) + r, \text{ where } r > 0$$

   Using the fact that $r > 0$, $a + c < a + c + r = b + c$:

   $$\therefore a + c < b + c$$

   □

2. Multiplying both sides of an inequality by a positive number preserves the inequality.

   Example. Suppose that $a = 4$ and $b = 16$. We see that $a < b$. If $c = 5$, show that $ac < bc$.

   $$4 < 16$$
   $$4(5) = 20 \quad \text{and} \quad 16(5) = 80$$
   $$20 < 80 \qquad \text{True}$$

   Example. Suppose that $a = -2$ and $b = 4$. We see that $a < b$. If $c = 3$, show that $ac < bc$.

   $$-2 < 4$$
   $$-2(3) = -6 \quad \text{and} \quad 4(3) = 12$$
   $$-6 < 12 \qquad \text{True}$$

Prove: If $a < b$, then, for some $c > 0, c \in R$, then $ac < bc$.

Proof. Since $a < b$ there exists $r > 0$ such that

$$b = a + r, \text{ where } r > 0$$

For equalities we can multiply both sides by the same number and the equality will still be true:

$$bc = (a + r)c$$
$$bc = ac + rc$$

Since $r > 0$ and $c > 0$ then $rc > 0$. Using the fact that rc is positive, we can conclude that:

$$ac < ac + rc = bc$$
$$\therefore ac < bc,\ \ c > 0,\ \ c \in R$$

□

3. Dividing both sides of an inequality by a positive number preserves the inequality.

Example. Suppose that $a = 4$ and $b = 16$. We see that $a < b$. If $c = 3$, show that $\frac{a}{c} < \frac{b}{c}$.

$$4 < 16$$
$$4 \div 3 = \frac{4}{3} \quad \text{and} \quad 16 \div 3 = \frac{16}{3}$$
$$\frac{4}{3} < \frac{16}{3} \qquad \text{True}$$

Example. Suppose that $a = -2$ and $b = 4$. We see that $a < b$. If $c = 2$, show that $\frac{a}{c} < \frac{b}{c}$.

$$-2 < 4$$
$$-2 \div 2 = -1 \quad \text{and} \quad 4 \div 2 = 2$$
$$-1 < 2 \qquad \text{True}$$

Prove: If $a < b$, then, for $c > 0, c \in R$, then $\frac{a}{c} < \frac{b}{c}$.

Proof.

$$b = a + r, \text{ where } r > 0$$
$$\frac{b}{c} = \frac{a + r}{c}$$
$$\frac{b}{c} = \frac{a}{c} + \frac{r}{c}$$

Since $r > 0$ and $c > 0$ then $\frac{r}{c} > 0$. Using the fact that $\frac{r}{c}$ is positive, we can conclude that:

$$\frac{a}{c} < \frac{a}{c} + \frac{r}{c} = \frac{b}{c}$$

$$\therefore \frac{a}{c} < \frac{b}{c},\ \ c > 0,\ \ c \in R$$

□

4. Multiplying both sides of an inequality by a negative number changes the direction of the inequality.

Example. Suppose that $a = 4$ and $b = 5$. We see that $a < b$. If $c = -2$, show that $ac > bc$.

$$4 < 5$$
$$4 \times (-2) = -8 \quad \text{and} \quad 5 \times (-2) = -10$$
$$-8 \nless -10$$
$$-8 > -10$$

Prove: If $a < b,\ c < 0,\ c \in R$, then $ac > bc$.

Proof. We can rewrite $a < b$ as:

$$b = a + r, \text{ where } r > 0$$

We can then multiply both sides of the equality by the same number and it will still be true:

$$bc = (a + r)c$$
$$bc = ac + rc$$

Since $r > 0$ and $c < 0$ then $rc < 0$. Using the fact that rc is negative, we can conclude that:

$$ac > ac + rc = bc$$

$$\therefore ac > bc,\ c < 0,\ c \in R$$

□

5. Dividing both sides of an inequality by a negative number changes the direction of the inequality.

Example. Suppose that $a = -6$ and $b = 9$. We see that $a < b$. If $c = -3$, show that $\frac{a}{c} > \frac{b}{c}$.

$$-6 < 9$$
$$-6 \div (-3) = 2 \quad \text{and} \quad 9 \div (-3) = -3$$
$$2 \nless -3$$
$$2 > -3$$

Prove: If $a < b,\ c < 0,\ c \in R$, then $\frac{a}{c} > \frac{b}{c}$.

Proof.

$$b = a + r, \text{ where } r > 0$$
$$\frac{b}{c} = \frac{a + r}{c}$$
$$\frac{b}{c} = \frac{a}{c} + \frac{r}{c}$$

Since $r > 0$ and $c < 0$ then $\frac{r}{c} < 0$. Using the fact that $\frac{r}{c}$ is negative, we can conclude that:

$$\frac{a}{c} > \frac{a}{c} + \frac{r}{c} = \frac{b}{c}$$

$$\therefore \frac{a}{c} > \frac{b}{c},\ c < 0,\ c \in R$$

□

6. If $0 < a < b$, then prove $a^2 < b^2$.

Proof.

$$\begin{aligned} b &= a + r \text{ where } r > 0 \\ b^2 &= (a+r)^2 \\ b^2 &= a^2 + \underbrace{2ar + r^2}_{>0} \end{aligned}$$

Since $a > 0$ and $r > 0$ then $2ar > 0$ and $r^2 > 0$. We can then conclude that $2ar + r^2 > 0$ and:

$$a^2 < a^2 + 2ar + r^2 = b^2$$

$$\therefore a^2 < b^2,\ 0 < a < b$$

□

7. If $0 < a < b$, then prove $\frac{1}{a} > \frac{1}{b}$. That is, for $0 < a < b$, the reciprocal of a will be greater than the reciprocal of b.

Proof.

$$\begin{aligned} b &= a + r \text{ where } r > 0 \\ \frac{b}{ab} &= \frac{a+r}{ab} \\ \frac{b}{ab} &= \frac{a}{ab} + \frac{r}{ab} \\ \frac{1}{a} &= \frac{1}{b} + \underbrace{\frac{r}{ab}}_{>0} \end{aligned}$$

Since $r > 0$, $a > 0$, and $b > 0$, then $\frac{r}{ab} > 0$. We can conclude that

$$\frac{1}{a} = \frac{1}{b} + \frac{r}{ab} > \frac{1}{b}$$

$$\therefore, \text{ for } 0 < a < b,\ \frac{1}{a} > \frac{1}{b}$$

□

Solving Inequalities Examples

1. Solve the inequality $-7 < -3x + 17 \leq 19$.

Solution 1: Separate the inequalities and work with the two sides separately.

$$\begin{array}{rcr} -7 < -3x + 17 & \text{and} & -3x + 17 \leq 19 \\ -24 < -3x & & -3x \leq 2 \\ \therefore 8 > x & \text{and} & x \geq \dfrac{-2}{3} \end{array}$$

$$\therefore \left\{ x \mid \frac{-2}{3} \leq x < 8,\ x \in \mathbb{R} \right\}$$

Solution 2: Apply this operations to each of the three parts at the same time.

$$\begin{array}{rcl} -7 < & -3x + 17 & \leq 19 \\ -7 - 17 < & -3x + 17 - 17 & \leq 19 - 17 \\ -24 < & -3x & \leq \ \ 2 \\ \dfrac{-24}{-3} > & \dfrac{-3x}{-3} & \geq \dfrac{2}{-3} \\ 8 > & x & \geq \ \dfrac{2}{-3} \end{array}$$

$$\therefore \left\{ x \mid \frac{-2}{3} \leq x < 8,\ x \in \mathbb{R} \right\}$$

2. Solve the inequality $9 + 4x \leq -3x + 2 \leq -x + 6$.

Solution: Split the inequality into parts.

$$\begin{array}{rcl} 9 + 4x & \leq & -3x + 2 \\ 4x + 3x & \leq & 2 - 9 \\ 7x & \leq & -7 \\ x & \leq & -1 \end{array} \quad \text{and} \quad \begin{array}{rcl} -3x + 2 & \leq & -x + 6 \\ -3x + x & \leq & 6 - 2 \\ -2x & \leq & 4 \\ x & \geq & -2 \end{array}$$

$$\therefore \{x \mid -2 \leq x \leq -1,\ x \in \mathbb{R}\}$$