Intermediate Math Circles
Wednesday November 12 2014
Problem Set Solutions

1. The point $(a, 2)$ is the point of intersection of the lines with equations $y = 2x - 4$ and $y = x + k$. Determine the value of k.

   Solution

   Since $(a, 2)$ is the point of intersection of the two lines, $(a, 2)$ satisfies $y = 2x - 4$, so $2 = 2(a) - 4$, $2a = 6$ and $a = 3$. The point $(3, 2)$ is on both lines.

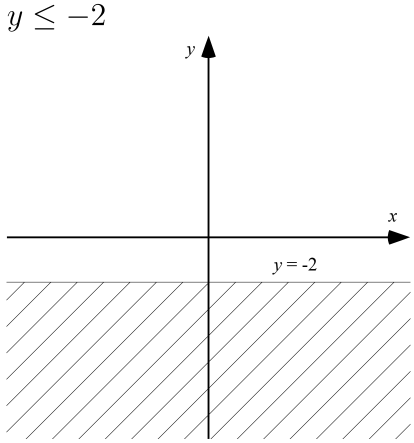
   Substitute $x = 3$, $y = 2$ into the equation $y = x + k$, to obtain $2 = 3 + k$. Hence $k = -1$.

2. Graph the following regions.

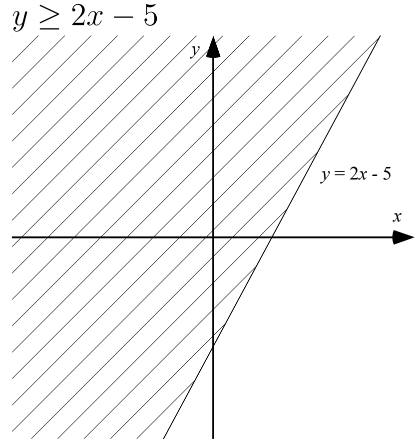
   a) $y \leq -2$ b) $x > 3$ c) $y \geq 2x - 5$ d) $2x + y < 4$

   Solution

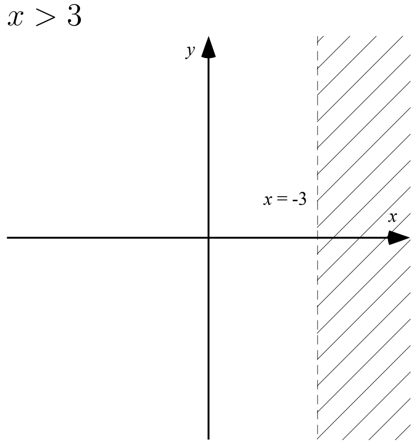
   a) $y \leq -2$



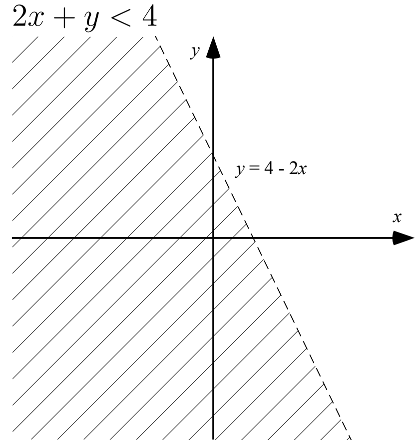
   c) $y \geq 2x - 5$



   b) $x > 3$



   d) $2x + y < 4$



   (**N. B.** To decide which side of the line to shade, choose a **test point** clearly not on the line. If it satisfies the inequality, shade in that side of the line. Otherwise, shade the other side. A formal check of a test point is provided in the solution to 3a. If the inequality is a strict inequality, the line should be dashed in the solution. Otherwise, the line is solid and is part of the solution.)

3. To find x-intercepts, set $y = 0$ and solve for x. To find y-intercepts, set $x = 0$ and solve for y. Graph the following regions by finding intercepts.

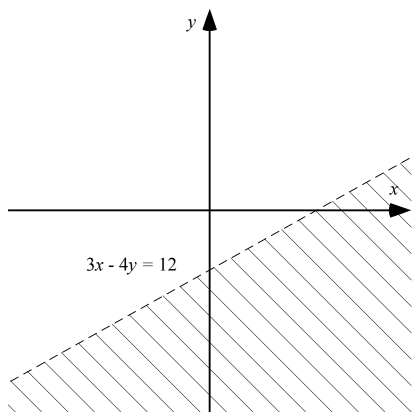
   a) $3x - 4y > 12$

   Solution

   x-intercept: Setting $y = 0$ in $3x - 4y = 12$ gives $3x = 12$ and $x = 4$ follows. Hence the x-intercept is $(4, 0)$.

   y-intercept: Setting $x = 0$ in $3x - 4y = 12$ gives $-4y = 12$ and $y = -3$ follows. Hence the y-intercept is $(0, -3)$.

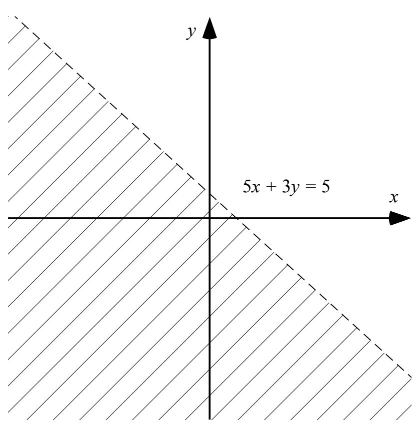
   Plotting the x- and y-intercepts, we can sketch the line. To see which side of the line to shade, choose the test point $(0, 0)$. The left side of the inequality is $3x - 4y = 3(0) - 4(0) = 0$. But the right side is 12, and clearly $0 \not> 12$. Hence $(0, 0)$ is NOT in the region. The line is dashed and the region is below the line.



   b) $5x + 3y \leq 5$

   Solution

   Setting $y = 0$ in $5x + 3y = 5$ gives $5x = 5$ and $x = 1$ follows. Hence the x-intercept is $(1, 0)$. Setting $x = 0$ in $5x + 3y = 5$ gives $3y = 5$ and $y = \frac{5}{3}$ follows. Hence the y-intercept is $(0, \frac{5}{3})$. Plotting the x- and y-intercepts, we can sketch the line. If $(0, 0)$ is the test point, the inequality is satisfied, meaning the origin is in the solution. Since it is not a strict inequality, the line is solid and the region is the area which includes the line and the region below it.



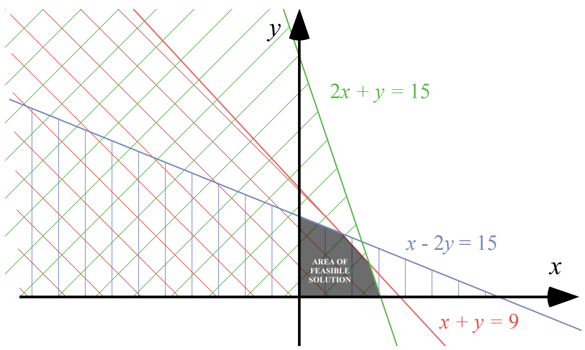
4(a) Graph the feasible region given the following inequalities:

$$x + y \leq 9$$
$$x + 2y \leq 15$$
$$2x + y \leq 15$$
$$x \geq 0$$
$$y \geq 0$$

Solution

Graph the inequalities using any method. For each line, use test points to figure out which side of the lines to shade. The feasible region is the area in which all points satisfy every inequality (essentially, where all the shadings overlap).

The grey shaded region is the feasible region. The problem did not ask you to find the points of intersection.



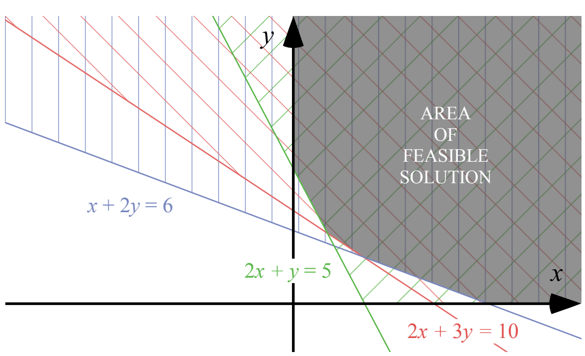
4(b) Graph the feasible region given the following inequalities:

$$x + 2y \geq 6$$
$$2x + y \geq 5$$
$$2x + 3y \geq 10$$
$$x \geq 0$$
$$y \geq 0$$

Solution

Graph the inequalities using any method. For each line, use test points to figure out which side of the lines to shade. The feasible region is the area in which all points satisfy every inequality (essentially, where all the shadings overlap).

The grey shaded region is the feasible region.



5. The formula for converting Celsius temperature C to a Fahrenheit tempeature F is given by $F = \frac{9}{5}C + 32$.

Andrew does not like arithmetic. So he approximates the Fahrenheit temperature by doubling C and then by adding 30 to get f, the approximate conversion.

If $f < F$, then the error in the approximation is $F - f$; otherwise, the error in the approximation is $f - F$. Determine the largest possible error in the approximation that Andrew would make when converting Celsius temperatures C with $20 \leq C \leq 35$.

Solution

Let $f = 2C + 30$ represent the approximate conversion. Let $F = \dfrac{9}{5}C + 32$ represent the actual conversion.

Consider the restriction $20 \leq C \leq 35$. Observe that within this interval of values for C, $f > F$. Then the error is $f - F = (2C + 30) - (\dfrac{9}{5}C + 32) = \dfrac{C}{5} - 2$.

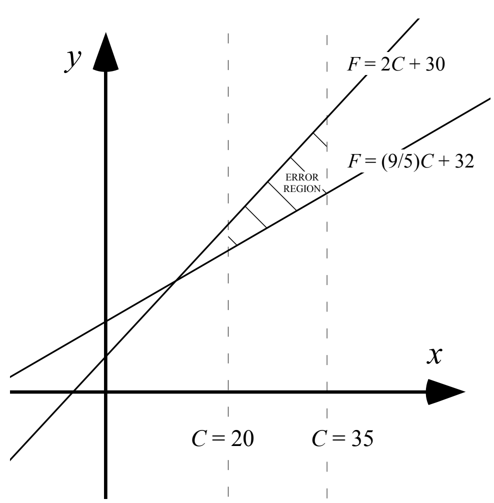
Manipulating the compound inequality gives

$$
\begin{aligned}
& & 20 \leq C \leq 35 \\
&\text{(Divide by 5)} & 4 \leq \frac{C}{5} \leq 7 \\
&\text{(Subtract 2)} & 2 \leq \frac{C}{5} - 2 \leq 5 \\
& & 2 \leq f - F \leq 5
\end{aligned}
$$

Hence the largest possible error is 5; Andrew's approximation will be off by at most 5 degrees.

Alternate Solution

Sketch the equations $F = \dfrac{9}{5}C + 32$, $f = 2C + 30$, $C = 20$ and $C = 35$ to obtain the graph below.



In the shaded area (error region), the maximum error is equal to the maximum separation within the region. Clearly, as C goes from 20 to 35, the error increases. So the maximum error occurs at the right endpoint, $C = 35$. The magnitude of this error is 5. Hence the largest possible error is 5 degrees.

6. Suppose that x and y are positive numbers with

$$xy = \frac{1}{9}$$
$$x(y+1) = \frac{7}{9}$$
$$y(x+1) = \frac{5}{18}$$

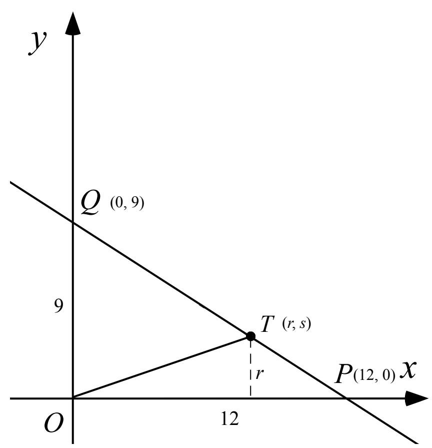
What is the value of $(x+1)(y+1)$?

Solution

Multiply the second and third equations together.

$$\begin{aligned} x(y+1)\cdot y(x+1) &= \frac{7}{9}\cdot\frac{5}{18} \\ (xy)(y+1)(x+1) &= \frac{7}{9}\cdot\frac{5}{18} \\ \frac{1}{9}(y+1)(x+1) &= \frac{7}{9}\cdot\frac{5}{18} \qquad (xy = \frac{1}{9}) \\ (y+1)(x+1) &= \frac{7}{9}\cdot\frac{5}{18}\cdot 9 \\ \therefore (x+1)(y+1) &= \frac{35}{18} \end{aligned}$$

7. The line $y = -\frac{3}{4}x + 9$ crosses the x-axis at P and the y-axis at Q. Point $T(r, s)$ is on line segment $\overline{PQ}$. If the area of $\triangle POQ$ is three times the area of $\triangle TOP$, then what is the value of $r + s$?



P and Q occur at the x and y intercepts respectively. When $x = 0$, the y-intercept occurs when $y = 9$. When $y = 0$, the x-intercept occurs when $x = 12$.

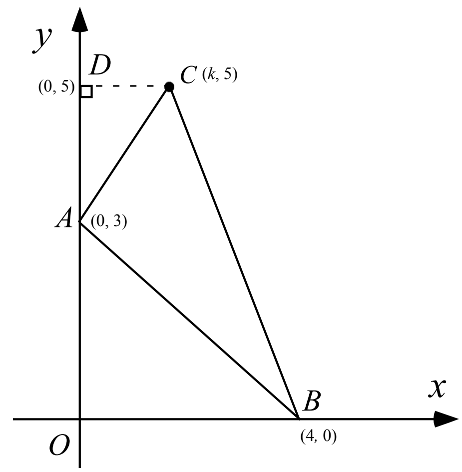
In the diagram, $\triangle QOP$ and $\triangle TOP$ share a common base. If $\triangle TOP$ is one-third the area of $\triangle QOP$, then the height of $\triangle TOP$ must be one third that of $\triangle QOP$ (since the bases are equal and area $= \frac{1}{2}bh$). So $r = \frac{1}{3}(9) = 3$.

Since $T = (r, s) = (3, s)$ lies on the line $y = -\dfrac{3}{4}x + 9$, $s = -\dfrac{3}{4}(3) + 9 = \dfrac{27}{4}$.

Thus $r + s = 3 + \dfrac{27}{4} = \dfrac{39}{4}$.

8. A triangle has vertices $A(0,3)$, $B(4,0)$, $C(k,5)$, where $0 < k < 4$. If the area of the triangle $\triangle ABC$ is 8, determine the value of k.

Solution



Construct DC parallel to OB. This forms a trapezoid $DOBC$ with height $OD = 5$, long side $OB = 4$ and short side $DC = k$.

The area of $DOBC$ is

$$\frac{1}{2}(OD)(DC + OB) = \frac{1}{2}(5)(4 + k) \qquad (1)$$

Observe that the trapezoid is composed of $\triangle CDA$, $\triangle CAB$ and $\triangle AOB$. So the area of the trapezoid is also

$$\text{Area}(\triangle AOB) + \text{Area}(\triangle CAB) + \text{Area}(\triangle CDA) \qquad (2)$$

The areas of the triangles are

- Area of $\triangle AOB = \frac{1}{2}(4)(3) = 6$
- Area of $\triangle CAB = 8$ (given)
- Area of $\triangle CDA = \frac{1}{2}(AD)(DC) = \frac{1}{2}(2)(k) = k$

Set (1) equal to (2) and solve for k

$$\begin{aligned}
\text{Area } DOBC &= 6 + 8 + k \\
\frac{1}{2}(5)(4 + k) &= 14 + k \\
5(4 + k) &= 28 + 2k \\
20 + 5k &= 28 + 2k \\
3k &= 8 \\
\therefore k &= \frac{8}{3}
\end{aligned}$$