Intermediate Math Circles
Wednesday November 12 2014
Equations and Inequalities with Two Variables

Problems from Last Week

10. What values of x satisfy the inequality $-3 < 5 - \frac{2}{x} < 3$? Sketch your solution.

Solution

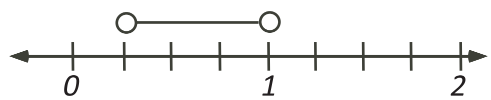
$$-3 < 5 - \frac{2}{x} < 3$$

$$-8 < -\frac{2}{x} < -2$$

Multiplying by -2,
$$4 > \frac{1}{x} > 1 \qquad (1)$$

Apply the reciprocal property to (1) $\frac{1}{4} < x < 1$.

Therefore the solution is $\frac{1}{4} < x < 1$.



13. If $\dfrac{\left(\frac{a}{c} + \frac{a}{b} + 1\right)}{\left(\frac{b}{a} + \frac{b}{c} + 1\right)} = 11$ where a, b, and c are positive integers, how many different ordered triples (a, b, c) are there such that $a + 2b + c \leq 40$ is true.

Solution

The solution will not be re-printed here.

It is available online at

http://www.cemc.uwaterloo.ca/events/mathcircle_presentations.html

Look at the link for solutions Nov 5/14 Inequality in the Intermediate section of Fall 2014.

It will also be available on tonight's video found at the same website.

Linear Equations With Two Variables

To talk about inequalities involving two variables, we need to develop some tools for equations in two variables. Linear equations in two variables can be represented geometrically as lines in the Cartesian plane $\mathbb{R}^2$, and can be written in the form:

$$y = mx + b$$

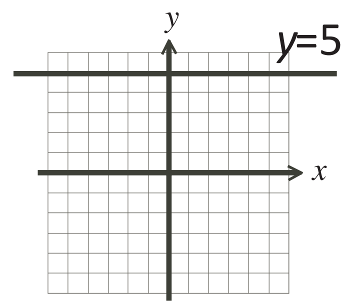
where $y, m, x, b \in \mathbb{R}$.

- y is the vertical position.
- m is the slope of the line.
- x is the horizontal position.
- b is the y intercept.

Graphing Lines

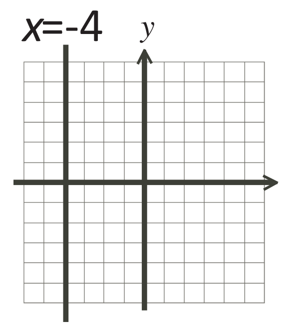
Given an equation of a line, we are interested in a geometric interpretation. We will look at some examples to see what happens.

Example 1: Sketch $y = 5$ in $\mathbb{R}^2$.



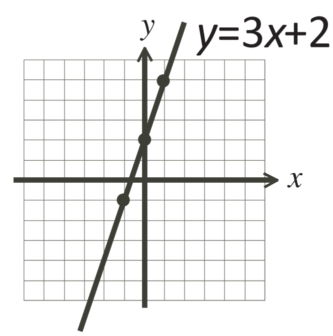
An equation of the form $y = b$ is a horizontal line b units above the x-axis if $b > 0$ and b units below the x-axis if $b < 0$. If $b = 0$, the horizontal line is on the x-axis. It follows that the equation of the x-axis is $y = 0$. In this specific example, the line is horizontal and 5 units above the x-axis.

Example 2: Sketch $x = -4$ in $\mathbb{R}^2$.



An equation of the form $x = a$ is a vertical line a units right of the y-axis if $a > 0$ and a units left of the y-axis if $a < 0$. If $a = 0$, the vertical line is on the y-axis. It follows that the equation of the y-axis is $x = 0$. In this specific example, the line is vertical and 4 units to the left of the y-axis.

Example 3: Sketch $y = 3x + 2$ in $\mathbb{R}^2$.



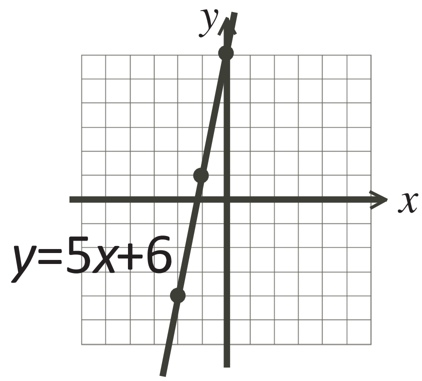
The line has y-intercept $b = 2$ so the point (0,2) is on the line. The line has slope $m = 3$. From any point on the line you can go up 3 units and right 1 unit OR go down 3 units and left 1 unit to move to other points on the line. Using this method, (1,5) and (-1, -1) are on the line. Draw the line so that it passes through the three points.

Example 4: Two distinct points define a line. Find the equation of the line that passes through $(-1, 1)$ and $(5, 31)$. Sketch the line.

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{31 - 1}{5 - (-1)} = \frac{30}{6} = 5$$

Now we have the equation $y = 5x + b$. To solve for the y-intercept we substitute the co-ordinates of a point on the line. In this case use the point $(-1, 1)$, so $x = -1$ and $y = 1$.

$$\begin{aligned} y &= 5x + b \\ 1 &= 5(-1) + b \\ 1 &= -5 + b \\ 6 &= b \end{aligned}$$



Therefore the equation of the line is $y = 5x + 6$. We can verify this by substituting $x = 5$ from the point $(5, 31)$ into $y = 5x + 6$. Since $5(5) + 6 = 31$, we obtain the correct y-coordinate. This verifies the correctness of the equation.

Systems of Equations:

Now we want to look at the intersection of two lines. There are three possibilities:

1. No intersections. This occurs when both lines have the same slope and different intercepts.
2. Infinite intersections. If the two equations represent the same line.
3. One intersection. If the above cases do not apply, then the lines intersect in a single point.

Solving for Intersections:

First check to see if the lines are parallel or the lines are multiples of each other. This can usually be done by inspection and will save you time. Otherwise we need to use the following methods:

1. **Substitution:** This is best used when x or y are already isolated.

   Example 5: Find the point of intersection of the lines:

   $$y = -x + 5 \qquad (1)$$
   $$y = 3x - 3 \qquad (2)$$

   Substitute the value of y from equation (1) into equation (2).

   $$\begin{aligned} -x + 5 &= 3x - 3 \\ -4x &= -8 \\ x &= 2 \end{aligned}$$

   Now determine the value of y by substituting $x = 2$ into equation (1).

   $$\begin{aligned} y &= -2 + 5 \\ y &= 3 \end{aligned}$$

   Therefore, the point of intersection is $(2, 3)$.

2. **Elimination:** Works nicely when lines are in standard form like $Ax + By = C$.

   Example 6: Find the point of intersection of the lines:

$$3x - 2y = -10 \qquad (1)$$
$$4x + 3y = -2 \qquad (2)$$

   We can eliminate y if we make the coefficients of y the same or opposite in both equations.

$$\begin{array}{llcll} (1) \times 3 & 9x - 6y & = & -30 & (3) \\ (2) \times 2 & 8x + 6y & = & -4 & (4) \\ (3)+(4) & 17x & = & -34 & \\ & x & = & -2 & \end{array}$$

   We can now substitute $x = -2$ into equation (1) to solve for y.

$$\begin{aligned} 3(-2) - 2y &= -10 \\ -6 - 2y &= -10 \\ -2y &= -4 \\ y &= 2 \end{aligned}$$

   Therefore, the point of intersection becomes $(-2, 2)$

Solving for Multiple Intersections:

What if we have more than two lines? In order to solve for all intersections, we take all possible combinations of pairs of lines and solve using either substitution or elimination.

Example 7: Solve for all intersections and graph the following system of equations:

$$y = -x \qquad (1)$$
$$y = x \qquad (2)$$
$$x + 2y = 12 \qquad (3)$$

First, we will use elimination for equations (1) and (2). We can eliminate x if we add the first two equations:

$$(1) + (2) \qquad y + y = -x + x$$
$$2y = 0$$
$$y = 0$$

We substitute $y = 0$ back into equation (1) to solve for the x-coordinate.

$$0 = -x$$
$$x = 0$$

Therefore, we have a point of intersection of (0,0) for lines (1) and (2).

Next we will solve for the intersection between (1) $y = -x$ and (3) $x + 2y = 12$. We can substitute $-x$ for y into equation (3).

$$x + 2(-x) = 12$$
$$-x = 12$$
$$x = -12$$

Substitute $x = -12$ back into equation (1) to solve for the y-coordinate.

$$y = -(-12) = 12$$

Therefore, the point of intersection for lines (1) and (3) is $(-12, 12)$.

Finally, we will solve for the intersection between (2) $y = x$ and (3) $x + 2y = 12$. Substitute x for y into equation (3).

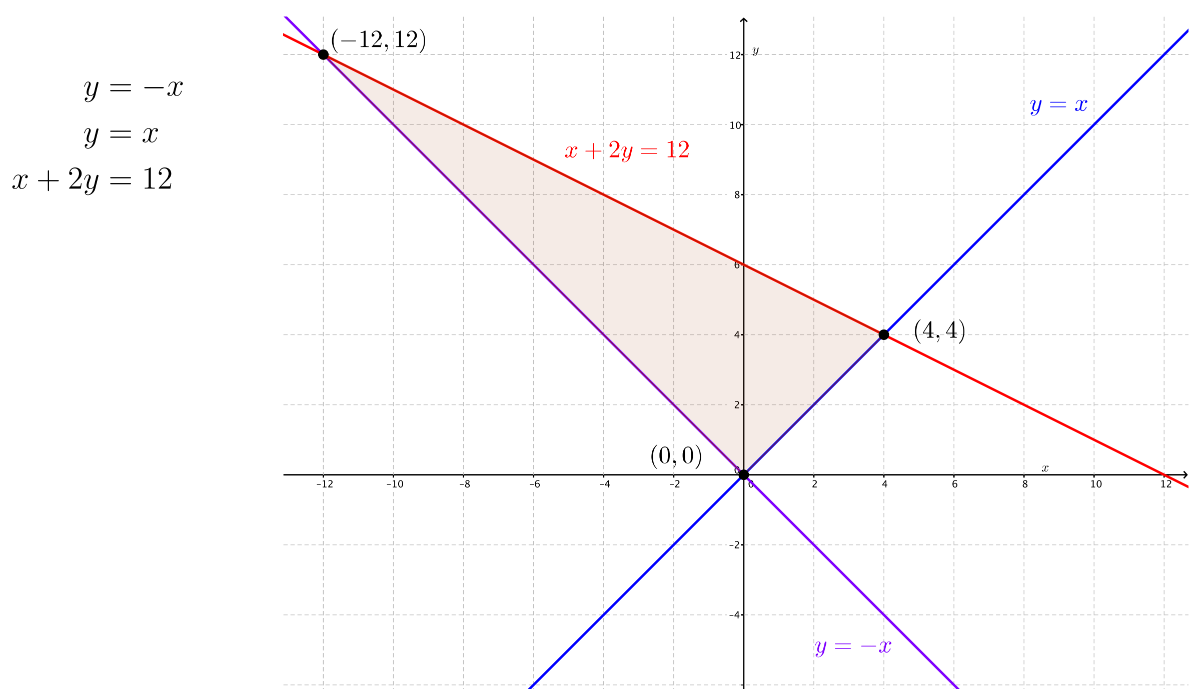
$$x + 2(x) = 12$$
$$3x = 12$$
$$x = 4$$

Substitute our value of x back in to equation (2).

$$y = 4$$

Therefore, the point of intersection for lines (2) and (3) is $(4, 4)$.

The completed graph is shown.



Inequalities in Two Variables:

Consider the line $x + y = 3$. This line divides the xy-plane into three regions:

1. Points that satisfy the equation $x + y = 3$.
2. Points that satisfy the inequality $x + y < 3$.
3. Points that satisfy the inequality $x + y > 3$.

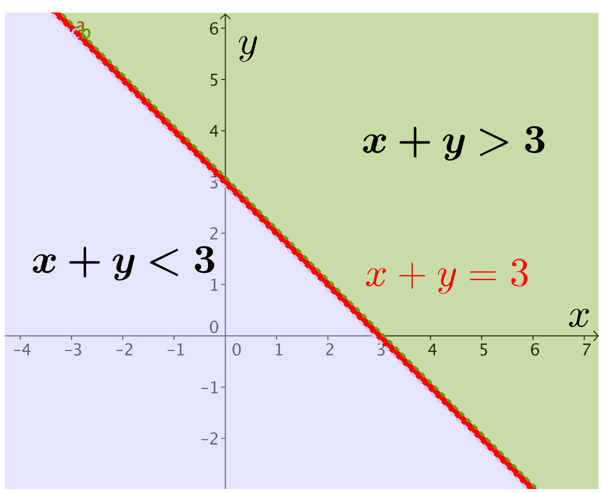
Example 8: Graph the inequality $x + y \leq 3$.

First, we will graph the line $x + y = 3$ using intercepts: the x-intercept is 3 and the y-intercept is 3.

Now pick a point that is clearly not on the line. This point is referred to as a test point.

- If the inequality is true when both sides are evaluated, then the test point is in the region and you would shade that part of the region.
- Otherwise, the point is not in the region and the region is on the other side of the line.

In this example, $(x, y) = (0, 0)$, the origin, is clearly not on the line.
$x + y = 0 + 0 = 0$ and since $0 < 3$, the test point $(0, 0)$ is in the region.
In this example, the solution includes all the points on the line and all points below the line.



If we are working with a strict inequality, the line is shown with a dashed line. For example, if we were graphing $x + y < 3$, we would dash the line $x + y = 3$ and shade below it.

System of Linear Inequalities

Now we begin to put it all together. In problems involving a system of inequalities, the solution of the system is the set of points that satisfy *all* the inequalities simultaneously.

Example 9: Graph the solution to the following inequalities:

$$\begin{aligned} 2x + 1.5y &\leq 90 \\ \frac{1}{2}x + \frac{1}{4}y &\leq 20 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Determine all points of intersection.

The area which contains the solution is called the *feasible region*. Hopefully, we will be able to discuss this further at another time.

Solution:

To graph $2x + 1.5y = 90$, determine the x- and y- intercepts. For the x-intercept, let $y = 0$. Then, $2x = 90$ and the x-intercept is 45. For the y-intercept, let $x = 0$. Then, $1.5y = 90$ and the y-intercept is 60. We can graph this line.

We want $2x + 1.5y \leq 90$. Checking with the graph, $(0, 0)$ is not on the line. Substituting $x = 0,\ y = 0$ into $2x + 1.5y$, we get 0 which is clearly < 90. Therefore, $(0, 0)$ is in the region to be shaded and we shade below the line. Since it is $\leq$, the line is solid.

To graph $\frac{1}{2}x + \frac{1}{4}y = 20$, determine the x- and y- intercepts. For the x-intercept, let $y = 0$. Then, $\frac{1}{2}x = 20$ and the x-intercept is 40. For the y-intercept, let $x = 0$. Then, $\frac{1}{4}y = 20$ and the y-intercept is 80. We can graph this line.

We want $\frac{1}{2}x + \frac{1}{4}y \leq 20$. Checking with the graph, $(0, 0)$ is not on the line. Substituting $x = 0,\ y = 0$ into $\frac{1}{2}x + \frac{1}{4}y$ we get 0 which is clearly < 20. Therefore, $(0, 0)$ is in the region to be shaded and we shade below the line. Since it is $\leq$, the line is solid.

Since $x \geq 0$ and $y \geq 0$, our solution only contains points in the first quadrant, the origin, points on the positive x-axis and points on the positive y-axis.

We also need to find the points of intersection.

$x = 0$ and $y = 0$ intersect at the origin.

$x = 0$ intersects $2x + 1.5y = 90$ at $(0, 60)$, the y-intercept and $x = 0$ intersects $\frac{1}{2}x + \frac{1}{4}y = 20$ at $(0, 80)$, the y-intercept.

$y = 0$ intersects $2x + 1.5y = 90$ at $(45, 0)$, the x-intercept and $y = 0$ intersects $\frac{1}{2}x + \frac{1}{4}y = 20$ at $(40, 0)$, the x-intercept.

Now we need to determine the point of intersection between (1) $2x + 1.5y = 90$ and (2) $\frac{1}{2}x + \frac{1}{4}y = 20$.

$$\begin{array}{lrcll} (1)\times 2 & 4x+3y & = & 180 & (3) \\ (2)\times 4 & 2x+y & = & 80 & (4) \end{array}$$

$$\begin{array}{lrcll} & 4x+3y & = & 180 & (3) \\ (4)\times 2 & 4x+2y & = & 160 & (5) \end{array}$$

$$\begin{array}{lrcl} (3)\text{-}(5) & y & = & 20 \\ \text{Subst. } y=20 \text{ into (2)} & \frac{1}{2}x+\frac{1}{4}(20) & = & 20 \\ & \frac{1}{2}x+5 & = & 20 \\ & \frac{1}{2}x & = & 15 \\ & x & = & 30 \end{array}$$

(1) $2x + 1.5y = 90$ and (2) $\frac{1}{2}x + \frac{1}{4}y = 20$ intersect at $(30, 20)$.