If you have been here before

- Check your name on the attendance sheet
- Pick up this week's handout

While you are waiting try this question.

Find the sum of the following series.
$22 + 23 + 24 + 25 + \cdots + 49 + 50$

We will start very close to 6:30.

Intermediate Math Circles November 26, 2014

Jeff Anderson

CIMC Solutions and Cool Questions

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November 26, 2014

Intermediate Math Circles - Night at a Glance

1. Look at some Math to do our Warmup Question.
2. Take up 2 CIMC Questions.
3. Look at some Brain Math to keep you sharp.

- Start promptly at 6:30, End 8:30, Break 10 minutes near 7:30
- Washrooms are located to the left and right.

Intermediate Math Circles - Reminders

Topics

Tonight is the last session for the Fall.
Math Circles will resume on February 4
Please sign the list on the table if you are not coming back in February.

Pascal, Cayley, Fermat Contests

February 24

Fryer, Galios, Hypatia

April 16

Johann Carl Friedrich Gauss

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3. Contributed to number theory, statistics, differential geometry, geophysics, electrostatics, optics and astronomy.
4. Recommended Sophie Germain to receive her honorary degree.
5. Has a CEMC math contest for Grade 7 and 8 students named after him.

But my favourite Gauss story is.....

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$$S = 1 + 2 + 3 + \cdots + 99 + 100$$

But my favourite Gauss story is.....

$$\begin{array}{rcrcrcrcccrcr} S & = & 1 & + & 2 & + & 3 & + & \cdots & + & 99 & + & 100 \\ S & = & 100 & + & 99 & + & 98 & + & \cdots & + & 2 & + & 1 \\ \hline \end{array}$$

But my favourite Gauss story is.....

$$
\begin{array}{rcrcrcrcccrcr}
S & = & 1 & + & 2 & + & 3 & + & \cdots & + & 99 & + & 100 \\
S & = & 100 & + & 99 & + & 98 & + & \cdots & + & 2 & + & 1 \\
\hline
2S & = & 101 & + & 101 & + & 101 & + & \cdots & + & 101 & + & 101
\end{array}
$$

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$$
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S & = & 1 & + & 2 & + & 3 & + & \cdots & + & 99 & + & 100 \\
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\hline
2S & = & 101 & + & 101 & + & 101 & + & \cdots & + & 101 & + & 101
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$$

$$2S = 100(101)$$

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S & = & 1 & + & 2 & + & 3 & + & \cdots & + & 99 & + & 100 \\
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\hline
2S & = & 101 & + & 101 & + & 101 & + & \cdots & + & 101 & + & 101
\end{array}$$

$$2S = 100(101)$$

$$S = \frac{100(101)}{2}$$

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\hline
2S & = & 101 & + & 101 & + & 101 & + & \cdots & + & 101 & + & 101
\end{array}
$$

$$
\begin{aligned}
2S &= 100(101) \\
S &= \frac{100(101)}{2} \\
S &= 5050
\end{aligned}
$$

More Gauss.

But then Gauss went further...

More Gauss.

<u>But then Gauss went further...</u>

$$S = 1 + 2 + \cdots + (n\text{-}1) + n$$

More Gauss.

But then Gauss went further...

$$\begin{array}{cccccccccccc} S & = & 1 & + & 2 & + & \cdots & + & (n\text{-}1) & + & n \\ S & = & n & + & (n\text{-}1) & + & \cdots & + & 2 & + & 1 \\ \hline \end{array}$$

More Gauss.

But then Gauss went further...

$$\begin{array}{rcrcrcccrcr} S & = & 1 & + & 2 & + & \cdots & + & (n\text{-}1) & + & n \\ S & = & n & + & (n\text{-}1) & + & \cdots & + & 2 & + & 1 \\ \hline 2S & = & n+1 & + & n+1 & + & \cdots & + & n+1 & + & n+1 \end{array}$$

More Gauss.

But then Gauss went further...

$$
\begin{array}{rcrcrcccrcr}
S & = & 1 & + & 2 & + & \cdots & + & (n-1) & + & n \\
S & = & n & + & (n-1) & + & \cdots & + & 2 & + & 1 \\
\hline
2S & = & n+1 & + & n+1 & + & \cdots & + & n+1 & + & n+1
\end{array}
$$

$$2S = n(n+1)$$

More Gauss.

But then Gauss went further...

$$\begin{array}{rcccccccccc} \text{S} & = & 1 & + & 2 & + & \cdots & + & \text{(n-1)} & + & \text{n} \\ \text{S} & = & \text{n} & + & \text{(n-1)} & + & \cdots & + & 2 & + & 1 \\ \hline \text{2S} & = & \text{n+1} & + & \text{n+1} & + & \cdots & + & \text{n+1} & + & \text{n+1} \end{array}$$

$$\text{2S} = \text{n(n+1)}$$

$$\text{S} = \frac{n(n+1)}{2}$$

More Gauss.

But then Gauss went further...

$$\begin{array}{ccccccccccc} S & = & 1 & + & 2 & + & \cdots & + & \text{(n-1)} & + & n \\ S & = & n & + & \text{(n-1)} & + & \cdots & + & 2 & + & 1 \\ \hline 2S & = & n+1 & + & n+1 & + & \cdots & + & n+1 & + & n+1 \end{array}$$

$$\begin{aligned} 2S &= n(n+1) \\ S &= \frac{n(n+1)}{2} \end{aligned}$$

The sum of the first n natural numbers is $\frac{n(n+1)}{2}$

Practice Problems

Find the sum of the natural numbers from 1 to 2014.

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Find the sum of the natural numbers from 1 to 2014.

$\frac{2014(2015)}{2} = 2029105$

Warmup Problem

Evaluate $22 + 23 + 24 + 25 + \cdots + 49 + 50$

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Evaluate $22 + 23 + 24 + 25 + \cdots + 49 + 50$
$= (1 + 2 + \cdots + 49 + 50) - (1 + 2 + \cdots + 20 + 21)$

Warmup Problem

Evaluate $22 + 23 + 24 + 25 + \cdots + 49 + 50$
$= (1 + 2 + \cdots + 49 + 50) - (1 + 2 + \cdots + 20 + 21)$
$= \frac{50(51)}{2} - \frac{21(22)}{2}$
$= 1275 - 231$
$= 1044$

Practice Problems

Find the sum of all multiples of 5 from 5 to 2015.
i.e. Sum $5 + 10 + 15 + \cdots + 2010 + 2015$

Practice Problems

Find the sum of all multiples of 5 from 5 to 2015.

i.e. Sum $5 + 10 + 15 + \cdots + 2010 + 2015$

$= 5(1 + 2 + 3 + \cdots + 402 + 403)$

$= 5 \times \frac{403(404)}{2}$

$= 407030$

Proof by Induction

1. First we show it works to start. i.e. Works for 1.
   LS=1 RS= $\frac{1(1+1)}{2} = 1$

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   i.e. $1 + 2 + 3 = \cdots + (k-1) + k = \frac{k(k+1)}{2}$

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   i.e. $1 + 2 + 3 = \cdots + (k-1) + k = \frac{k(k+1)}{2}$
3. Use the assumption to prove it works for $k+1$.
   i.e. $1 + 2 + 3 = \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$

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   i.e. $1 + 2 + 3 = \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$
   LS$= 1 + 2 + 3 = \cdots + k + (k+1)$
   $= (1 + 2 + 3 = \cdots + k) + (k+1)$

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   LS$= 1 + 2 + 3 = \cdots + k + (k+1)$
   $= (1 + 2 + 3 = \cdots + k) + (k+1)$
   $= \frac{k(k+1)}{2} + (k+1)$
   $= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$
   $= \frac{k^2+3k+2}{2}$
   $= \frac{(k+1)(k+2)}{2} = RS$

Therefore we have proven that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

Try these 3 versions.

1. $1 + 2 + 3 + \cdots + 149 + 150 + 200 + 201 + 202 + \cdots + 299 + 300$
2. $9 + 12 + 15 + \cdots + 99 + 102$
3. $4 + 7 + 10 + \cdots + 298 + 301$

Question 1

$$1 + 2 + 3 + \cdots + 149 + 150 + 200 + 201 + 202 + \cdots + 299 + 300$$

Question 1

$$1 + 2 + 3 + \cdots + 149 + 150 + 200 + 201 + 202 + \cdots + 299 + 300$$
$$= (1 + 2 + 3 + \cdots + 299 + 300) - (151 + 152 + \cdots + 198 + 199)$$

Question 1

$$1 + 2 + 3 + \cdots + 149 + 150 + 200 + 201 + 202 + \cdots + 299 + 300$$
$$= (1 + 2 + 3 + \cdots + 299 + 300) - (151 + 152 + \cdots + 198 + 199)$$
$$= (\frac{300(301)}{2}) - ((1 + 2 + \cdots + 198 + 199) - (1 + 2 + \cdots + 149 + 150))$$

Question 1

$$1 + 2 + 3 + \cdots + 149 + 150 + 200 + 201 + 202 + \cdots + 299 + 300$$
$$= (1 + 2 + 3 + \cdots + 299 + 300) - (151 + 152 + \cdots + 198 + 199)$$
$$= (\tfrac{300(301)}{2}) - ((1 + 2 + \cdots + 198 + 199) - (1 + 2 + \cdots + 149 + 150))$$
$$= 45150 - (\tfrac{199(200)}{2} - \tfrac{150(151)}{2})$$

Question 1

$$
\begin{aligned}
&1+2+3+\cdots+149+150+200+201+202+\cdots+299+300\\
&=(1+2+3+\cdots+299+300)-(151+152+\cdots+198+199)\\
&=(\tfrac{300(301)}{2})-((1+2+\cdots+198+199)-(1+2+\cdots+149+150))\\
&=45150-(\tfrac{199(200)}{2}-\tfrac{150(151)}{2})\\
&=45150-(19900-11325)\\
&=36575
\end{aligned}
$$

Question 2

$$9 + 12 + 15 + \cdots + 99 + 102$$

Question 2

$$9 + 12 + 15 + \cdots + 99 + 102$$
$$= 3(3 + 4 + 5 + \cdots + 33 + 34)$$

Question 2

$$
\begin{aligned}
&9 + 12 + 15 + \cdots + 99 + 102 \\
&= 3(3 + 4 + 5 + \cdots + 33 + 34) \\
&= 3((1 + 2 + 3 + 4 + 5 + \cdots + 33 + 34) - (1 + 2))
\end{aligned}
$$

Question 2

$$
\begin{aligned}
&9 + 12 + 15 + \cdots + 99 + 102 \\
&= 3(3 + 4 + 5 + \cdots + 33 + 34) \\
&= 3((1 + 2 + 3 + 4 + 5 + \cdots + 33 + 34) - (1 + 2)) \\
&= 3(\tfrac{34(35)}{2} - 3)
\end{aligned}
$$

Question 2

$$\begin{aligned}
&9 + 12 + 15 + \cdots + 99 + 102 \\
&= 3(3 + 4 + 5 + \cdots + 33 + 34) \\
&= 3((1 + 2 + 3 + 4 + 5 + \cdots + 33 + 34) - (1 + 2)) \\
&= 3(\tfrac{34(35)}{2} - 3) \\
&= 1776
\end{aligned}$$

Question 3

$$4 + 7 + 10 + \cdots + 298 + 301$$

Question 3

$$4 + 7 + 10 + \cdots + 298 + 301$$
$$= (3 + 1) + (6 + 1) + (9 + 1) + \cdots + (297 + 1) + (300 + 1)$$

Question 3

$$4 + 7 + 10 + \cdots + 298 + 301$$
$$= (3 + 1) + (6 + 1) + (9 + 1) + \cdots + (297 + 1) + (300 + 1)$$
$$= (1\times3+1)+(2\times3+1)+(3\times3+1)+\cdots+(99\times3+1)+(100\times3+1)$$

Question 3

$$
\begin{aligned}
&4 + 7 + 10 + \cdots + 298 + 301 \\
&= (3 + 1) + (6 + 1) + (9 + 1) + \cdots + (297 + 1) + (300 + 1) \\
&= (1\times3+1)+(2\times3+1)+(3\times3+1)+\cdots+(99\times3+1)+(100\times3+1) \\
&= 3 \times (1 + 2 + 3 + \cdots + 99 + 100) + 100 \times 1
\end{aligned}
$$

Question 3

$$
\begin{aligned}
&4 + 7 + 10 + \cdots + 298 + 301 \\
&= (3 + 1) + (6 + 1) + (9 + 1) + \cdots + (297 + 1) + (300 + 1) \\
&= (1\times 3+1)+(2\times 3+1)+(3\times 3+1)+\cdots+(99\times 3+1)+(100\times 3+1) \\
&= 3 \times (1 + 2 + 3 + \cdots + 99 + 100) + 100 \times 1 \\
&= 3 \times 5050 + 100 \\
&= 15250
\end{aligned}
$$

CIMC A6

A6. A positive integer is a prime number if it is greater than 1 and has no positive divisors other than 1 and itself.

The integer 43797 satisfies the following conditions:

- each pair of neighbouring digits (read from left to right) forms a two-digit prime number, and
- all of the prime numbers formed by these pairs are different.

What is the largest positive integer that satisfies both of these conditions?

A6 solution

Start with the two digit primes:
11,13,17,23,29,31,37,41,43,53,59,61,67,71,73,79,83,89,97

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11,13,17,23,29,31,37,41,43,53,59,61,67,71,73,79,83,89,97
Split them into the not too useful primes with an even digit and the very useful totally odd primes.
Even: 23,29,41,43,61,67,83,89
Odd: 11,13,17,19,31,37,53,59,71,73,79,97

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So something like 311371973 or 8971373119 would be good numbers.

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So something like 311371973 or 8971373119 would be good numbers.
In the odd group which digit is most common?
Which even goes on the front?

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So something like 311371973 or 8971373119 would be good numbers.
In the odd group which digit is most common?
Which even goes on the front?
And the answer is......
619737131179

CIMC B1

1a. Determine the average of the six integers 22,23,23,25,26,31.

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The average of the 6 integers given is

$$\frac{22+23+23+25+26+31}{6} = \frac{150}{6} = 25.$$

1b. the average of the three numbers $y+7, 2y-9, 8y+6$ is 27. What is the value of y?

1b. the average of the three numbers $y + 7, 2y - 9, 8y + 6$ is 27. What is the value of y?

Since the average of the three numbers $y + 7$, $2y - 9$ and $8y + 6$ is 27, then the sum of the three numbers is $3(27) = 81$.

Therefore, $(y + 7) + (2y - 9) + (8y + 6) = 81$ or $11y + 4 = 81$, and so $11y = 77$ or $y = 7$.

1c. Four positve integers, not necessarily different and each less than 100, have an average of 94. Determine, with explanation, the minimum possible value for one of these integers.

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Since the average of four integers is 94, then their sum is $4(94) = 376$.
Since the sum of the integers is constant, then for one of the integers to be as small as possible, the other three integers must be as large as possible.

1c. Four positve integers, not necessarily different and each less than 100, have an average of 94. Determine, with explanation, the minimum possible value for one of these integers.
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Then we should let the other 3 integers be 99 as that is the largest allowed.

1c. Four positve integers, not necessarily different and each less than 100, have an average of 94. Determine, with explanation, the minimum possible value for one of these integers.
Since the average of four integers is 94, then their sum is $4(94) = 376$.
Since the sum of the integers is constant, then for one of the integers to be as small as possible, the other three integers must be as large as possible.
Then we should let the other 3 integers be 99 as that is the largest allowed.
Then $99 + 99 + 99 + x = 376$
$x = 79$

One thing we are always trying to improve on to become better Mathemeticians is to have active minds that see patterns well

www.krazydad.com

This is a wonderful site for all sorts of puzzles.