Intermediate Math Circles
Wednesday 15 October 2014
Geometry II: Side Lengths

Last week we discussed various angle properties. As we progressed through the evening, we proved many results. This week, we will look at various side length properties and we will prove some results. Some of this material will be familiar and some of this will stretch what you already know.

Problems From Last Week

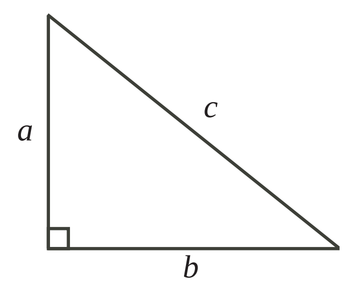
Let us take up three problems from last week. We will have volunteers present a solution for a prize. Complete solutions can be found on our website at
http://www.cemc.uwaterloo.ca/events/mathcircle_presentations.html.

Remark from Last Week's Lesson

Our proof that the interior angles of an n-gon sum to $180°(n-2)$ only works if each interior angle is less than $180°$.
We call these types of polygons: ________________________.

Getting Started

The Pythagorean Theorem:



In a right-angled triangle, the *hypotenuse* is the longest side and is located opposite the $90°$ angle. In any right-angled triangle, the square of the hypotenuse equals the sum of the squares of the other two sides.

In the triangle illustrated to the right, ________________________.

Proofs of The Pythagorean Theorem:

If you do an internet search you will discover many different proofs of the Pythagorean Theorem. If you go to the link http://www.cut-the-knot.org/pythagoras/index.shtml#84, you will find 98 of the proofs grouped together. We will present two proofs here and a third one will presented online in the completed note for tonight's session.

Proof #1:

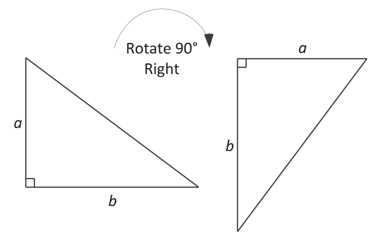
The first proof presented is a visual proof.

Proof #2:

In the note that will be published online, a second proof is presented.

Proof #3:

Starting with the leftmost right triangle, rotate 90° to the right to create the second triangle.



This proof is attributed to James Garfield, the twentieth President of the United States.

A *Pythagorean Triple* is a triple (a, b, c) of positive integers with $a^2+b^2 = c^2$. What Pythagorean Triples do you know?

The chart illustrates several Pythagorean triples. The smallest side length is an odd number.

a	b	c
3	4	5
5	12	13
7	24	25
9	40	41
11	60	61

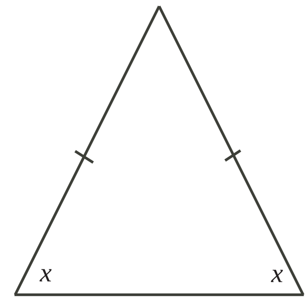
Look for patterns in the table.

Can you predict the triple in which the smallest number is 13?

Can you predict a formula for generating any Pythagorean Triple with a, the smallest number, an odd number ≥ 3.

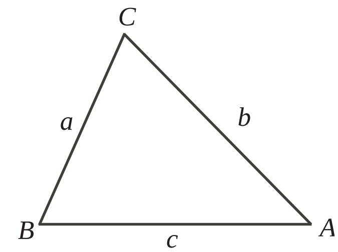
Proof:

If a triangle has two angles equal, then the two opposite sides are equal.
That is, the triangle is isosceles.



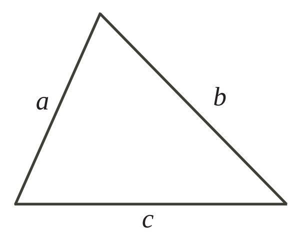
The Isosceles Triangle Theorem

Relating Angles and Sides



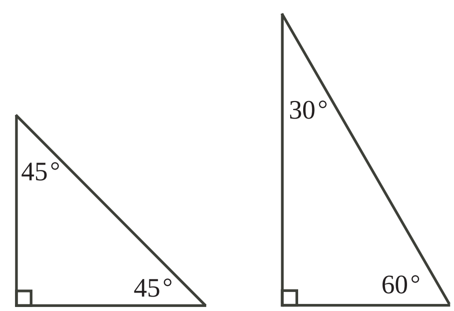
Triangle Inequality Law

If a, b and c are the side lengths of a triangle, the *Triangle Inequality* tells us that $b + c > a$ and $a + c > b$ and $a + b > c$.
Can you explain why this is true?



Special Triangles

There are two kinds of special triangles.
The first has angles 45°, 45° and 90°.
The second has angles 30°, 60° and 90°.
If the shortest side in each has length 1, what are the other side lengths? (These can be scaled by any factor.)



Congruent Triangles

Two triangles are called *congruent* if corresponding side lengths and corresponding angles are all equal. In other words, the triangles are equal in all respects. Sometimes, fewer than these 6 equalities are necessary to establish congruence. Four ways to determine that two triangles are congruent:

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-
-
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Once two triangles are proved to be congruent, all of the other corresponding equalities follow.

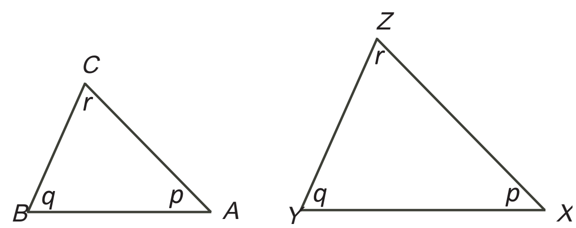
Similar Triangles

Two triangles are called *similar* if corresponding angles are equal.

• If two triangles are similar, then the corresponding pairs of sides are in a constant ratio.

Given: $\angle A = \angle X$ and $\angle B = \angle Y$ and $\angle C = \angle Z$.

Therefore,



In other words, the triangles are "scaled models" of each other.

• Two triangles are also similar if two pairs of corresponding sides are in constant ratio and the angles between the sides are equal.

Once similarity is shown then the remaining pair of corresponding sides are in a constant ratio and the other corresponding angles are equal.

Show that the triangles are similar.