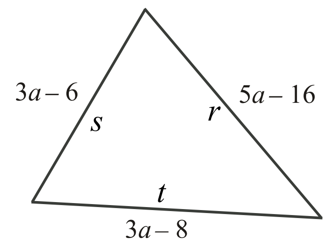
Intermediate Math Circles
Wednesday 15 October 2014
Problem Set 2 Solutions

1. Determine the number of different values of a for which the given triangle is isosceles.



Solution

A triangle is isosceles if at least two sides are equal.

Label the sides as shown in the diagram. There are three cases.

(a) If $r = s$

$$5a - 16 = 3a - 6$$
$$2a = 10$$
$$a = 5$$

So $a = 5$ is a solution.

(b) If $r = t$

$$5a - 16 = 3a - 8$$
$$2a = 8$$
$$a = 4$$

So $a = 4$ is a solution.

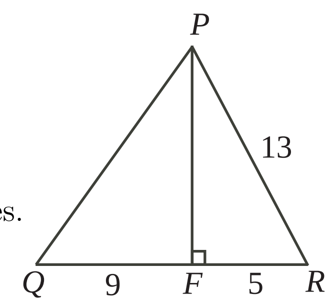
(c) If $s = t$

$$3a - 6 = 3a - 8$$
$$0a = 2$$

No solution.

Therefore, there are 2 values for a, $\{4, 5\}$, for which the triangle is isosceles.

2. In triangle PQR, F is the point on QR so that PF is perpendicular to QR. If $PR = 13$, $RF = 5$, and $FQ = 9$, what is the perimeter of $\triangle PQR$?



Solution

$\triangle PFR$ is right angled. Apply the Pythagorean Theorem to its sides.

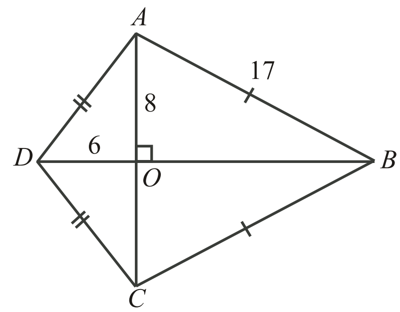
$$PF^2 + FR^2 = PR^2$$
$$PF^2 + 5^2 = 13^2$$
$$PF^2 = 13^2 - 5^2$$
$$PF^2 = 144$$
$$PF = 12 \qquad (PF > 0)$$

Now apply the Pythagorean Theorem to $\triangle PFQ$.

$$PQ^2 = PF^2 + FQ^2$$
$$PQ^2 = 12^2 + 9^2$$
$$PQ^2 = 225$$
$$PQ = 15 \qquad (PQ > 0)$$

Therefore, the perimeter is $PQ + QR + PR = 15 + 9 + 5 + 13 = 42$.

3. Calculate the area of figure $ABCD$.



Solution

Observe that since $AD = DC$, $AB = BC$ and $\triangle ADB$ and $\triangle CDB$ share a common side DB, $\triangle ADB \cong \triangle CDB$ by SSS.

Since $\triangle ADB \cong \triangle CDB$, Area $\triangle ADB$ = Area $\triangle CDB$.

Then Area $ADCB$ = Area $\triangle ADB$ + Area $\triangle CDB = 2 \times$ Area $\triangle ADB$

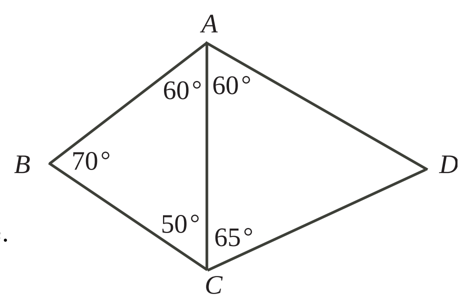
In $\triangle AOB$, $OB^2 = AB^2 - AO^2 = 17^2 - 8^2 = 225$, so $OB = \sqrt{225} = 15$.

Then $DB = 6 + 15 = 21$.

$$\begin{aligned}\text{Area of } \triangle ADB &= \frac{1}{2} \times DB \times OA \\ &= \frac{1}{2} \times 21 \times 8 \\ &= 84\end{aligned}$$

$$\begin{aligned}\therefore \text{Area of } ADCB &= 2 \times 84 \\ &= 168 \text{ units}^2\end{aligned}$$

4. In the diagram, which side is the longest: AB, BC, AC, CD, or AD?



Solution

In any triangle, the longest side subtends the largest angle.

In $\triangle ABC$, $\angle B > \angle A > \angle C$, so $AC > BC > AB$.

In $\triangle ADC$, $\angle ADC = 180 - 60 - 65 = 55°$.

Then $\angle C > \angle A > \angle D$. Therefore, $AD > CD > AC$.

Combining these inequalities produces $AD > CD > AC > BC > AB$.

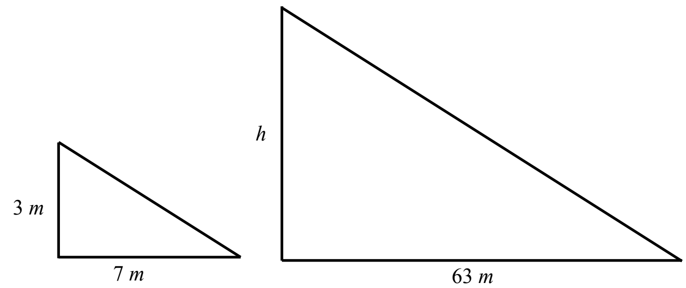
$\therefore AD$ is the longest side length in the diagram.

5. If a 3 m stake casts a shadow 7 m long, what is the height of a tree that casts a shadow 63 m long?

Solution

The left triangle in the diagram illustrates the given situation, while the right triangle illustrates the unknown.

The problem assumes the stake keeps the same orientation and that the sun is at the same point in the sky.



Then the two triangles are similar,

$$\therefore \quad \frac{h}{63} = \frac{3}{7}$$
$$7h = 3 \times 63$$
$$7h = 189$$
$$h = 27$$

$\therefore$ the tree has a height of 27 m.

6. A *scalene* triangle is a triangle whose side lengths are all different. Determine the side lengths of all possible scalene triangles with integer side lengths and perimeter less than 13.

Solution

Let $c,\ b,\ a$ be the side lengths of the triangle from smallest to largest, with $c < b < a$.

Notice that if $c \geq 4$, then $b \geq 5$ and $a \geq 6$. Summing the inequalities gives $a+b+c \geq 15$, i.e. a perimeter of 15, which is not allowed. So $1 \leq c \leq 3$.

Suppose $c = 1$. Since $b < a$, and these are all integers, it is also true that $b + 1 \leq a$. But we assumed $c = 1$ and so $b + c \leq a$; this is impossible by the side length condition for triangles. So $c \neq 1$.

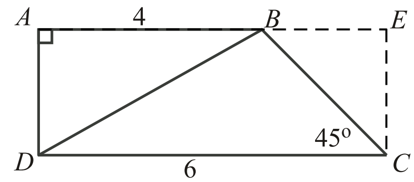
Suppose $c = 2$. If $b \geq 5$, then $a \geq 6$. Adding these inequalities gives $a+b+c = a+b+2 \geq 6+5+2 = 13$ - a perimeter of at least 13. This is not allowed. Therefore, $2 = c < b < 5$, or $b = 3, 4$.

For $b = 3$, if $5 \leq a \leq 6$, then $5 = c + b \leq a$, which is not possible by the side length condition. So $a < 5$; since $3 = b < a$, $a = 4$. Thus $a = 4$ is the only possibility when $b = 3$. For $b = 4$, if $a \geq 6$, then $a \geq 6 = 4 + 2 = b + c$, again not allowed. Thus $a = 5$ is the only possibility when $b = 4$. This gives two possibilities: $(a, b, c) = (4, 3, 2)$ and $(5, 4, 2)$.

Suppose $c = 3$. So $b > c > 3$. If $b \geq 5$, then $a \geq 6$, and so the perimeter will be greater than 12. Therefore, $3 < b < 5$, or $b = 4$. For $b = 4$, $a = 5$ is the only possibility (if $a = 6$, the total perimeter will exceed 13). This gives $(a, b, c) = (5, 4, 3)$.

Therefore, there are 3 possibilities in total: $(a, b, c) = (5, 4, 3)$ or $(5, 4, 2)$ or $(4, 3, 2)$.

7. In the diagram, $AB = 4$, $DC = 6$, and AB is parallel to DC. If $\angle C = 45^\circ$, determine the length of BD.



Solution

Construct $CE \parallel DA$ such that CE intersects AB extended at E.

Since $AB \parallel DC$, it follows that $AE \parallel DC$.

Applying the co-interior angle axiom, $\angle BAD + \angle ADC = 180^\circ$. But $\angle BAD = 90^\circ$, so $\angle ADC = 90^\circ$ follows.

Similarly, since $CE \parallel DA$, $\angle DAE + \angle CEA = 180^\circ$. But $\angle DAE = 90^\circ$, so $\angle CEA = 90^\circ$.

In $ABCD$, three of the angles are 90°. Since the sum of the angles in a quadrilateral is 360°, then the fourth angle, $\angle DCE$, is 90°.

Since $AE \parallel DC$, $\angle EBC = \angle DCB = 45^\circ$. Since $\angle DCE = 90^\circ$ and $\angle DCB = 45^\circ$, $\angle BCE = 45^\circ$. Then in $\triangle EBC$, $\angle EBC = \angle BCE = 45^\circ$. $\therefore BE = EC$.

In rectangle $ADCE$,

$$\begin{aligned} AE &= DC \\ AB + BE &= DC \\ 4 + BE &= 6 \\ BE &= 2 \\ \therefore EC &= 2 \qquad (BE = EC) \end{aligned}$$

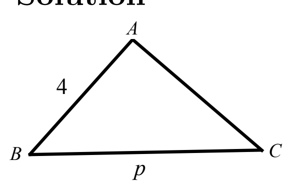
In rectangle $ADCE$, $AD = EC = 2$.

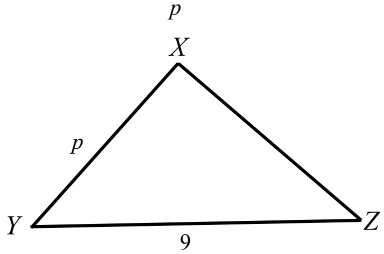
Applying the Pythagorean Theorem in $\triangle ADB$,

$$\begin{aligned} BD^2 &= AD^2 + AB^2 \\ &= 2^2 + 4^2 \\ BD &= 20 \qquad (BD > 0) \\ \therefore BD &= 2\sqrt{5} \end{aligned}$$

8. $\triangle ABC$ is similar to $\triangle XYZ$. If $AB = 4$, $YZ = 9$, and $BC = XY = p$, determine the value of p.

Solution





Both triangles are similar, so $\dfrac{AB}{XY} = \dfrac{BC}{YZ}$.

Substituting the knowns and unknowns and solving gives

$$\begin{aligned} \frac{4}{p} &= \frac{p}{9} \\ 36 &= p^2 \\ \therefore 6 &= p \end{aligned}$$

9. Using the side lengths 2, 3, 5, 7 and 11, how many different triangles *with exactly two equal sides* can be formed?

Solution

Let a be the length of the identical sides, and b the third side. Note $b \neq a$, otherwise three sides equal.

(a) Case 1: $a = 2$. Need $a + a > b$. The only valid choice is $b = 3$

(b) Case 2: $a = 3$, $2a = 6$. The only valid choices for b are $b = 2, 5$

(c) Case 3: $a = 5$, $2a = 10$. The only valid choices for b are $b = 2, 3, 7$

(d) Case 4: $a = 7$, $2a = 14$. The only valid choices for b are $b = 2, 3, 5, 11$

(e) Case 5: $a = 11$, $2a = 22$. The only valid choices for b are $b = 2, 3, 5, 7$

Therefore, there are $1 + 2 + 3 + 4 + 4 = 14$ different triangles possible.

10. A triangle can be formed having side lengths 4, 5 and 8. It is impossible however, to construct a triangle with side lengths 4, 5 and 10. Ron has eight sticks, each having an integer length. He observes that he cannot form a triangle using any three of these sticks as side lengths. What is the shortest possible length of the longest of the eight sticks?

Solution

The strategy for this problem is to start off with the smallest side lengths possible, and then figure out what the bigger side lengths must be.

Let two sticks be size 1. The third stick must have length $\geq 1 + 1 = 2$. So choose it to be length 2.

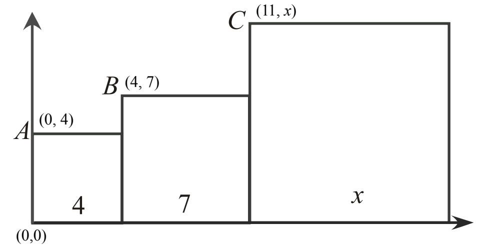
Choose the stick of length 1 and 2. The fourth stick has to have length greater than the maximum possible sum of the other sticks, i.e. it must be $\geq 1 + 2 = 3$, so it has length 3.

The fifth then must be of length $\geq 2 + 3 = 5$, so it has length 5.

The pattern used to generate the next stick is obvious - this is just the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$. The eighth term in this sequence is the longest stick, which is 21.

Therefore the longest stick length is 21.

11. In the adjacent squares shown, the vertices A, B and C lie in a straight line. What is the value of x?



Solution

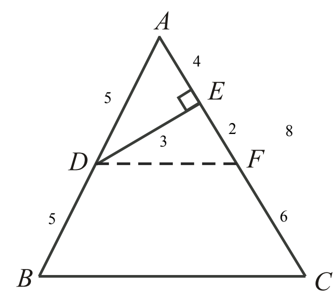
Draw the squares on a coordinate grid as shown.

Let $A = (0, 4)$, $B = (4, 7)$, $C = (11, x)$.

As these 3 points lie on a straight line, the slope of the line between any two must be the same. In particular,

$$\begin{aligned}\text{Slope AB} &= \text{Slope BC}\\ \frac{7-4}{4-0} &= \frac{x-7}{11-4}\\ \frac{3}{4} &= \frac{x-7}{7}\\ 21 &= 4x - 28\\ 49 &= 4x\\ \therefore \frac{49}{4} &= x\end{aligned}$$

12. In the diagram, $AD = BD = 5$, $EC = 8$ and $AE = 4$. Determine the length of BC.



Solution

Draw the line DF so that F is the midpoint of AC.

Since $AC = AE + EC = 8 + 4 = 12$, $AF = FC = 6$.

Then $EF = AF - AE = 6 - 4 = 2$. $\angle AED = 90°$ so $\angle DEF = 90°$.

Hence $\triangle DEF$ is right angled. By Pythagorean Theorem, in $\triangle AED$, $DE = 3$. Then in $\triangle DEF$, $DF = \sqrt{9+4} = \sqrt{13}$.

Observe that between $\triangle ADF$ and $\triangle ABC$, $\dfrac{AD}{AB} = \dfrac{5}{10} = \dfrac{1}{2}$, and $\dfrac{AF}{AC} = \dfrac{6}{12} = \dfrac{1}{2}$.

Since these two triangles also share a common angle $\angle A$, $\triangle ADF \sim \triangle ABC$.

Then $\dfrac{DF}{BC} = \dfrac{\sqrt{13}}{BC} = \dfrac{1}{2}$ and therefore $BC = 2\sqrt{13}$.