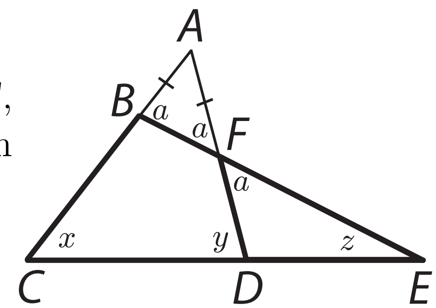
Intermediate Math Circles
Wednesday October 22 2014
Geometry III: Circles

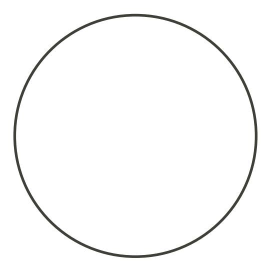
Looking back at Problem Set 1 Question 11

In the figure shown, $AB = AF$ and ABC, AFD, BFE, and CDE are all straight lines. Determine an equation relating x, y and z.

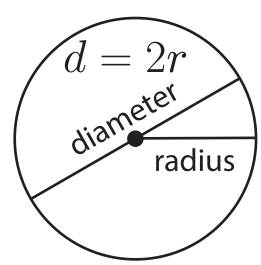


In $\triangle ABF$, since $AB = AF$ then $\angle ABF = \angle AFB = a$.
Since $\angle AFB$ and $\angle DFE$ are opposite angles, then $\angle AFB = \angle DFE = a$.
$\angle ABE$ is exterior to $\triangle BCE$. Therefore, $a = x + z$. (1)
$\angle CDF$ is exterior to $\triangle DFE$. Therefore, $y = a + z$. (2)
Substitute for a from (1) into (2). Therefore, $y = x + z + z$.
Rearranging, $x - y + 2z = 0$.

What do we know about circles?

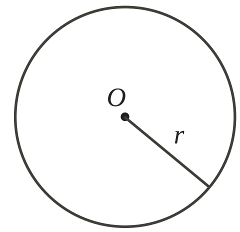


- Circles are round.
- Diameter $= 2\times$ radius.
- $A = \pi r^2$
- $C = \pi d = 2\pi r.$

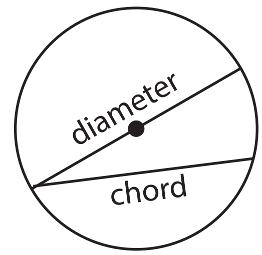


Definitions

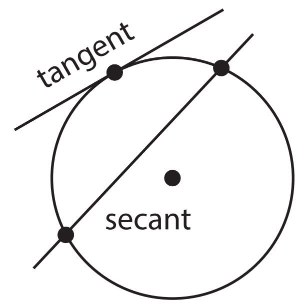
- A *circle* is a set of points in 2-space that are all equidistant from a fixed point. The fixed distance is called the *radius* and the fixed point is called the *centre*.



- A *chord* is a line segment with its endpoints on the circumference of a circle.



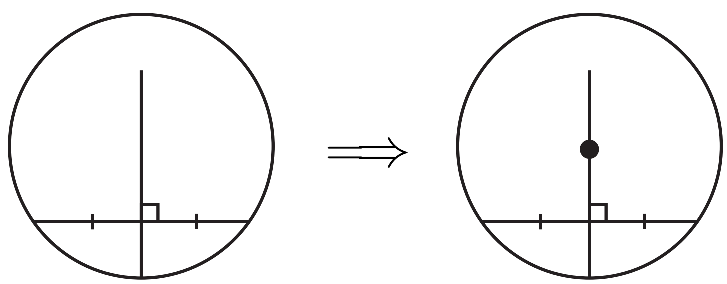
- A *diameter* is a chord that passes through the centre of a circle.
- A *tangent* is a line (or line segment) that touches a circle in exactly one point.



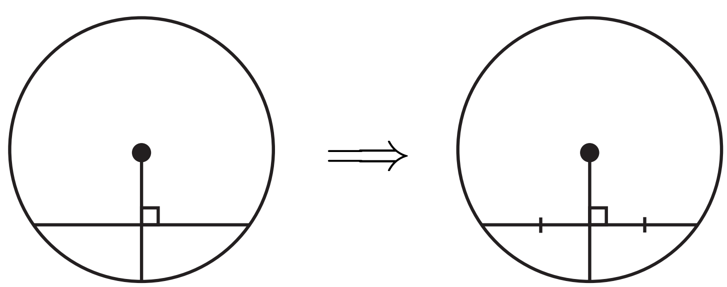
- A *secant* is a line that intersects a circle in two points.

Chord Right Bisector Property

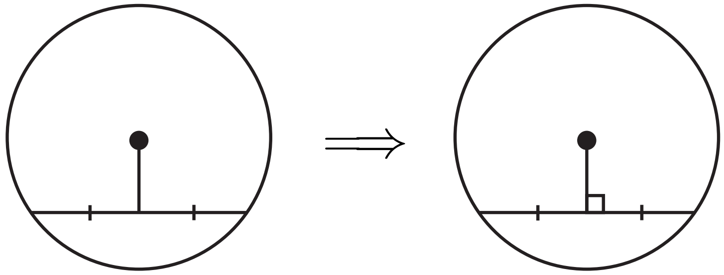
- The right bisector of a chord passes through the centre of the circle.



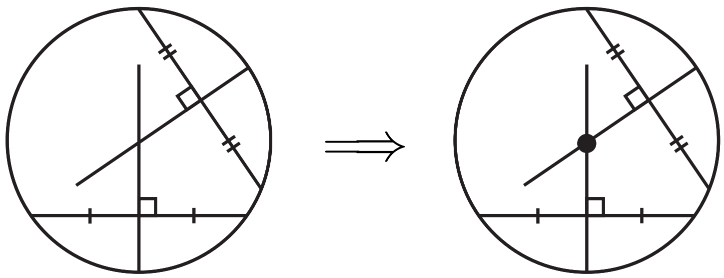
- The perpendicular from the centre to a chord bisects the chord.



- The line joining the centre to the midpoint of a chord is perpendicular to the chord.

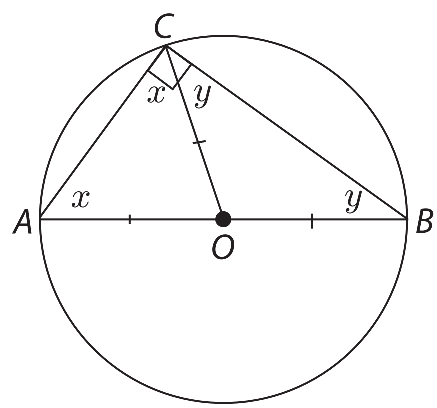


- The centre of a circle is the intersection of the right bisectors of two non-parallel chords.



Prove: The angle inscribed in a semi-circle is 90°.

Proof:



Draw a circle with centre O, diameter AB, and C another point on the circumference of the circle. Join O to C.

In $\triangle OAC$, both OA and OC are radii of the circle. Therefore, $\angle OAC = \angle OCA = x$.

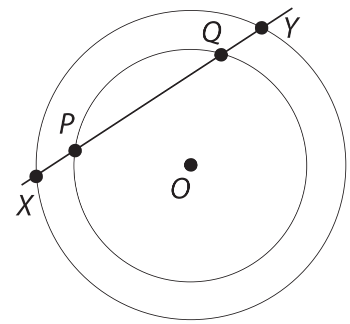
In $\triangle OBC$, both OB and OC are radii of the circle.
Therefore, $\angle OBC = \angle OCB = y$.
Since the angles in a triangle sum to 180°,

$$\begin{aligned} \angle BAC + \angle BCA + \angle ABC &= 180^\circ \\ x + (x + y) + y &= 180^\circ \\ 2x + 2y &= 180^\circ \\ x + y &= 90^\circ \end{aligned}$$

But $\angle ACB = x + y$. Therefore, $\angle ACB$, the angle inscribed in the semi-circle, is 90°.

Example 1:

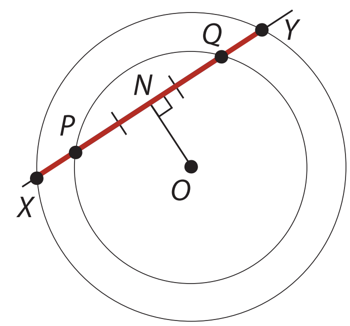
Two circles with the same centre are called *concentric circles*. A line is drawn through two concentric circles intersects the circles at X, P, Q, and Y, as shown. O is the centre of both circles.



Prove that $PX = QY$.

Proof:

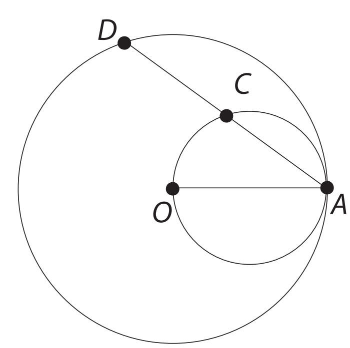
Construct a perpendicular from O to the line containing X, P, Q, Y, intersecting the line at N.
PQ is a chord of the smaller circle. ON is a perpendicular to PQ and passes through the centre. Therefore, $PN = NQ$. (1)
XY is a chord of the larger circle. ON is a perpendicular to XY and passes through the centre. Therefore, $XN = NY$. (2)



$$\begin{aligned} (2) - (1) \qquad XN - PN &= NY - NQ \\ PX &= QY, \quad \text{follows.} \end{aligned}$$

Example 2:

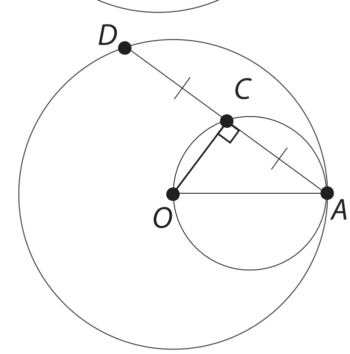
In the diagram, the two circles are tangent at A. If AO is the diameter of the smaller circle and the radius of the larger circle, prove that $AC = CD$.



Proof:

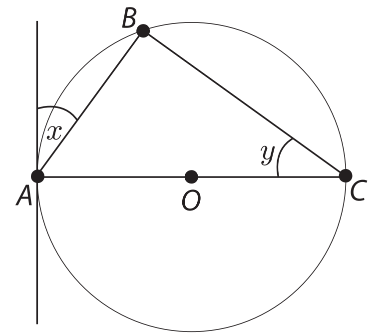
Join OC. $\angle OCA$ is inscribed in a semi-circle since OA is a diameter of the smaller circle. Therefore, $\angle OCA = 90°$.

But O is the centre of the larger circle with $OC \perp AD$. Therefore, C bisects chord AD and $AC = CD$ follows.



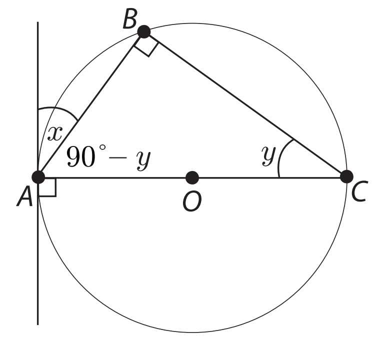
Example 3:

In the diagram, the line at A is tangent to the circle centred at O. Prove $x = y$.



Proof:

AC is a diameter of the circle, thus $\angle ABC = 90^\circ$. The tangent at A is perpendicular to the diameter, AC. Therefore, $\angle BAC = 90^\circ - y$. Also, $x + \angle BAC = 90^\circ$. Therefore,



$$\begin{aligned} x + (90^\circ - y) &= 90^\circ \\ x - y &= 90^\circ - 90^\circ \\ x - y &= 0 \\ x &= y \end{aligned}$$

Giving the desired result.