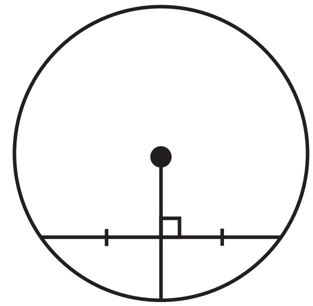
Intermediate Math Circles
Wednesday October 29 2014
Geometry IV: More Circle Geometry

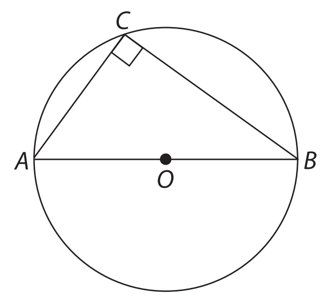
Recap from last Week

The Chord Right Bisector Property



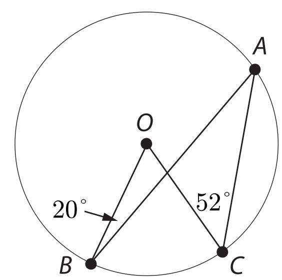
- The right bisector of a chord passes through the centre of the circle.
- The perpendicular from the centre to a chord bisects the chord.
- The line joining the centre to the midpoint of a chord is perpendicular to the chord.
- The centre of a circle is the intersection of the right bisectors of two non-parallel chords.

Angle in a Semicircle Property



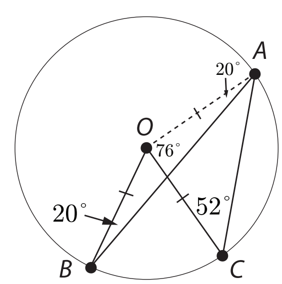
- An angle inscribed in a semicircle is 90°. In this case, if AB is a diameter, then $\angle ACB = 90°$.

Example 1



A circle with centre O has points A, B and C on its circumference. $\angle OBA = 20°$ and $\angle OCA = 52°$. Determine the measure of $\angle BOC$ and the measure of $\angle BAC$. What is the relationship between these two angles?

Solution



Join O to A. Since OB and OA are radii, $OB = OA$ and $\triangle OBA$ is isosceles. Therefore, $\angle OAB = \angle OBA = 20°$. It follows that $\angle BOA = 180° - 2 \times 20° = 140°$.

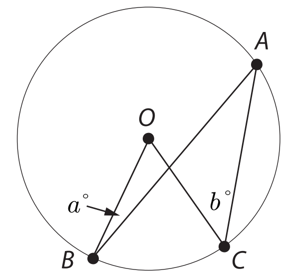
Since OC and OA are radii, $OC = OA$ and $\triangle OCA$ is isosceles. Therefore, $\angle OAC = \angle OCA = 52°$. It follows that $\angle COA = 180° - 2 \times 52° = 76°$.

Then, $\angle BOC = \angle BOA - \angle COA = 140° - 76° = 64°$ and $\angle BAC = \angle OAC - \angle OAB = 52° - 20° = 32°$.

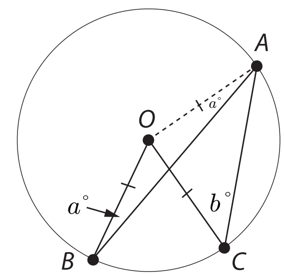
Since $\angle BAC = 32°$ and $\angle BOC = 64°$, $\angle BOC = 2 \times \angle BAC$.

Example 2

A circle with centre O has points A, B and C on its circumference. $\angle OBA = a^\circ$ and $\angle OCA = b^\circ$. Determine the measure of $\angle BOC$ and the measure of $\angle BAC$. What is the relationship between these two angles?



Solution



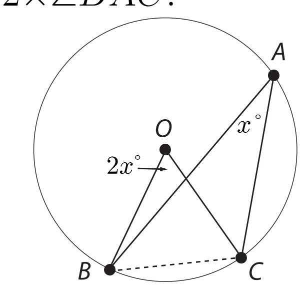
Join O to A. Since OB and OA are radii, $OB = OA$ and $\triangle OBA$ is isosceles. Therefore, $\angle OAB = \angle OBA = a^\circ$. It follows that $\angle BOA = (180^\circ - 2a)^\circ$.

Since OC and OA are radii, $OC = OA$ and $\triangle OCA$ is isosceles. Therefore, $\angle OAC = \angle OCA = b^\circ$. It follows that $\angle COA = (180^\circ - 2b)^\circ$.

Then, $\angle BOC = \angle BOA - \angle COA = (180^\circ - 2a)^\circ - (180^\circ - 2b)^\circ = (2b - 2a)^\circ$ and $\angle BAC = \angle OAC - \angle OAB = (b - a)^\circ$.

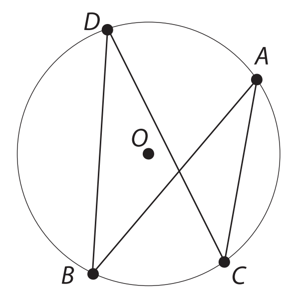
Since $\angle BAC = (b-a)^\circ$ and $\angle BOC = (2b-2a)^\circ = 2(b-a)^\circ$, $\angle BOC = 2\times\angle BAC$.

Angle at the circumference Property



An angle at the centre of a circle is twice the angle at the circumference standing on the same (side of a common) chord.

Angles Inscribed in a Circle by a Common Chord



Two angles inscribed in a circle and standing on the same side of a common chord are equal.

Proof:

From the diagram we are to show that $\angle BDC = \angle BAC$ since they are standing on the common chord BC. We could also show that $\angle DBA = \angle DCA$ since they are standing on the common chord DA. We will show the first result.

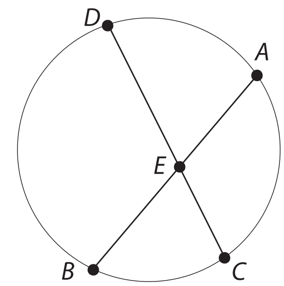
Using the angle at the circumference property,

$$\angle BOC = 2 \times \angle BDC \text{ and } \angle BOC = 2 \times \angle BAC.$$

Since $\angle BOC = \angle BOC$, then

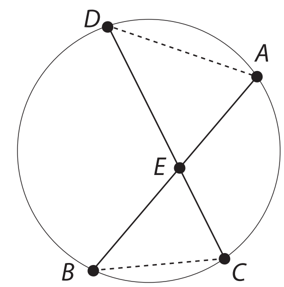
$$\begin{aligned} 2 \times \angle BDC &= 2 \times \angle BAC \\ \therefore \angle BDC &= \angle BAC \end{aligned}$$

Chord Splitting Property



If two chords in a circle intersect, the product of the lengths of the two parts of one chord is equal to the product of the lengths of the two parts of the other chord. In this case, prove $DE \times EC = AE \times EB$.

Proof:

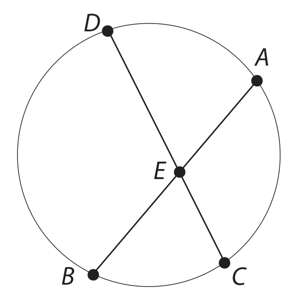


Join B to C and A to D. Since $\angle DAB$ and $\angle DCB$ share a common chord, $\angle DAB = \angle DCB$. Also, $\angle DEA = \angle BEC$ since they are vertically opposite angles. Therefore, $\triangle DAE \sim \triangle BCE$, using AA $\triangle \sim$.

From the similarity, $\dfrac{AE}{EC} = \dfrac{DE}{EB}$.

"Cross-multiplying", we obtain the required result $DE \times EC = AE \times EB$.

Example 3



Given $BE = EA$, $CE = 3$ and $DE = 12$. Determine the length of AB.

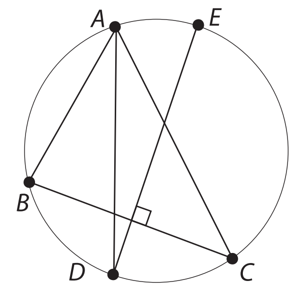
Solution

Let $BE = AE = x$. Using the chord splitting property,

$$\begin{aligned} BE \times EA &= CE \times ED \\ x \times x &= 3 \times 12 \\ x^2 &= 36 \\ x &= 6, \quad x > 0 \\ \text{Then, } AB &= AE + EB \\ AB &= 2x \\ AB &= 12 \end{aligned}$$

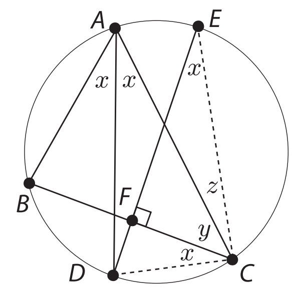
Example 4

$\triangle ABC$ has its vertices on a circle, as shown. The bisector of the angle at A meets the circumference at D. From D, a line is drawn perpendicular to the chord BC so that it meets the circumference at E. Prove that DE is a diameter of the circle.



Solution

If we can show that $\angle DCE = 90°$ then DE is a diameter of the circle.



Let F be the point where BC meets DE at $90°$.
Let $\angle BCA = y$ and $\angle ACE = z$.

Since DA bisects $\angle BAC$, $\angle BAD = \angle DAC = x$.

Since $\angle DAC$ and $\angle DEC$ are standing on the same chord DC, $\angle DEC = \angle DAC = x$.
Since $\angle BAD$ and $\angle BCD$ are standing on the same chord BD, $\angle BCD = \angle BAD = x$.

In right triangle EFC, $\angle FEC + \angle FCE = 90°$. Therefore, $x + y + z = 90°$.
But $\angle DCE = x + y + z$ and $\angle DCE = 90°$ follows.

Since $\angle DCE$ is a right angle inscribed in the circle by DE, it follows that DE is the diameter.

Want More?

Go to the Euclid Contest eWorkshop. There is a circle geometry package that will take you further with circles. Go to

http://cemc.math.uwaterloo.ca/contests/euclid_eWorkshop.html

and work through the material.