UNIVERSITY OF WATERLOO

Faculty of Mathematics
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Centre for Education in
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Introduction to Game Theory

Roger and Colleen have been placed in two separate interrogation rooms as suspects for breaking the vending machine in DC (**this event did not actually happen**). They are both guilty, but before committing the crime they agreed that they would both stay quiet if they were to get caught. However, all communication has been lost between the two of them and they are beginning to doubt how much trust they have in one another. The table represents the set of 4 possible outcomes that can occur depending on whether Roger and Colleen choose to confess to the crime or stay quiet.

Colleen

Roger		Confess	Stay Quiet
	Confess	5,5	0,10
	Stay Quiet	10,0	2,2

The rows of this table represent Roger's two decisions he can make, and the columns represent the two decisions Colleen can make. Combining their two decisions, we end up in one quadrant(section) of the table where the number in red corresponds to the length of Roger's detention sentence, and the number in blue corresponds to the length of Colleen's detention sentence. For example, if Roger decides to confess, then we are only dealing with outcomes in the first row of the table. If Colleen then chooses to stay quiet by choosing the second column, Roger will serve 0 hours in detention and Colleen will serve 10 (When they confess, they are confessing that both of them committed the crime).

Making a Decision...

Roger and Colleen did have an agreement to both stay quiet, but now there is too much at risk for them to trust one another. They must consider what the other person might do to make sure they get the best possible outcome for themselves. Assuming now that Roger wants to minimize his detention time, he must consider the following:

- If Colleen is to confess, then Roger's best decision is to **confess** (5 hours in detention is better than 10). Let's circle this outcome in red in the table below.
- If Colleen is to stay quiet, then Roger's best decision is to **confess** (0 hours in detention is better than 2). Let's circle this outcome in red in the table below.

Looking at Colleen's point of view:

- If Roger is to confess, then Colleen's best decision is to **confess**. Let's mark this outcome with blue brackets in the table below (just so we know whose point of view we are considering).
- If Roger is to stay quiet, then Colleen's best decision is to **confess**. Let's mark this outcome with blue brackets in the table below.

Colleen

Roger		Confess	Stay Quiet
	Confess	[5,5]	0,10
	Stay Quiet	[10,0]	2,2

You may think, "Why wouldn't they both stay quiet?", as it would lead to a better outcome than if they were to both confess. However, if Roger decides to keep their agreement and Colleen decides to break it, then Roger is going to serve 10 hours in detention. Therefore, by using this same logic for Colleen, choosing to **confess** is the best(safest) option for both Roger and Colleen.

Game Theory - The Prisoner's Dilemma

Colleen

Roger

	Confess	Stay Quiet
Confess	[5,5]	0,10
Stay Quiet	10,0	2,2

This famous example is known as the "Prisoners Dilemma," and is often referred to when talking about Game Theory. Game Theory is the study of deciding on the best strategy to win a game, and applies to games where the decision of one person affects the decision of the other. When we say "win a game," this can also be thought of as just satisfying our interests as much as we can, given a certain situation (game). In the "Prisoner's Dilemma" example, to the win the game meant to get the least amount of detention time as possible.

Game Theory is useful because it allows us to visually represent the possible outcomes that can occur when making decisions in certain situations. We can then check to see what the best decision is based on the decisions of our opponents, just like we did in the "Prisoner's Dilemma" example. This outcome we ended up with, where Roger and Colleen would both confess, is called a Nash Equilibrium in Game Theory.

Nash Equilibrium

The Nash Equilibrium is the outcome in a game where one player cannot become better off if they individually change their decision. Referring to our game above, the Nash Equilibrium lies in the first row and first column of the table, because Roger or Colleen would only become worse off if either of them switched their decision and changed the outcome (facing 10 hours in detention instead of 5). So, choosing this decision is the safest option for both Roger and Colleen. We will discover the method of finding the Nash Equilibrium for games that can be represented in a table. But first, let's take a look at another game!

Bonus Mark Game

There will be a test at the end of 8 weeks to see how well your math skills have developed over the course of the term (this was a joke to make the game as realistic as possible in class). Luckily for you, Math Circles is offering up bonus marks! However, instead of just giving you bonus marks, you're going to earn these bonus marks by playing a game. The rules are as follows:

- On the count of three you can choose to raise your hand or not.
- If everyone raises their hand, the whole class gets two bonus marks.
- If at least one person does not raise their hand, then everyone that did receives 0 bonus marks.
- If you don't raise your hand, you are guaranteed one bonus mark.

Rest of Class

		Hand Up	Hand Down
You	Hand Up	2,2	0,1
	Hand Down	1,0	1,1

Let's find out the best decisions you can make based on the decision of the rest of the class:

- If the rest of the class puts their hand up, then your best decision is to **put your hand up** (2 bonus marks is better than 1).
- If the rest of the class keeps their hand down, then your best decision is to **keep your hand down** (1 bonus mark is better than 0).

From the rest of the classes point of view (assuming the rest of the class acts as one person, making one decision):

- If you put your hand up, then the best decision for the rest of the class is to **put their hand up**.
- If you keep your hand down, then the best decision for the rest of the class to **keep their hand down**.

Rest of Class

You

	Hand Up	Hand Down
Hand Up	[2,2]	0,1
Hand Down	1,0	[1,1]

After repeating the same steps we took in the "Prisoner's Dilemma" game, notice that we have two Nash Equilibriums for this game! From each of these outcomes, neither player can individually change their decision to get a better outcome. In particular, if the rest of the class has their hand up, you would not want to keep your hand down, because then you would only receive 1 bonus mark instead of 2. Also, if the rest of the class keeps their hand down, you would not want to then put your hand up, because then you would receive 0 bonus marks instead of 1.

In the case that their are two Nash Equilibriums, we have think about what is good about making either decision and what is bad about making either decision, given the rules of the game. When we played this game in class, about half of the class put their hands up, while the other half kept their hands down. The students who put their hands up argued that if their is a chance to receive 2 bonus marks, which is more than 1, then everyone should simply just put their hands up. However, given the rules of the game, since all it took was for one student to keep their hand down and cause the students who put their hands up to receive 0 bonus marks, there was too much risk in trying to get the 2 bonus marks. Therefore, although there may be two Nash Equilibrium, or 2 "best" decisions for all players to make, you must consider the rules of the game while trying to make a decision.

What if the rule was, "If half of the class keeps their hand down, the students who put their hands up receive 0 bonus marks". Does this change your thinking?

Best Response Method - Pure Strategy

Here are the rules for the game below:

- The row player will choose a row and the column player will choose a column.
- Combining these two choices, we land on one square in the table.
- Positive, red values are payouts, in dollars, to the row player.
- Negative, blue values are payouts, in dollars, to the column player (-2 is pays the column player $2).
- All payouts are made by the losing player (loser pays the winner).

To find the best strategy for this game, we will use the "Best Response Method," which is deciding on the row that gives us the most satisfaction, based on the column that our opponent chose (same steps we took in the "Prisoner's Dilemma" and "Bonus Mark" game). We then repeat these steps for the column player, deciding on the column that gives them the most satisfaction, based on the row that we chose:

Column

	0	6	-2	-4
Row	5	2	1	3
	-8	-1	0	20

The row player's best response given the column's player decisions:

- If the column player chooses column 1, then the row player should choose **row 2**.
- If the column player chooses column 2, then the row player should choose **row 1**.
- If the column player chooses column 3, then the row player should choose **row 2**.
- If the column player chooses column 4, then the row player should choose **row 3**.

These are the outcomes that would occur from the work we did above:

Column

	0	6	-2	-4
Row	5	2	1	3
	-8	-1	0	20

Repeating these steps from the point of view of the column player:

- If the row player chooses row 1, then the column player should choose **column 4**.
- If the row player chooses row 2, then the column player should choose **column 3**.
- If the row player chooses row 3, then the column player should choose **column 1**.

Marking the outcomes that would occur we have:

Column

	0	6	2	[4]
Row	5	2	[1]	3
	[8]	1	0	20

After performing the "Best Response" method for each player, notice that one of the payouts was the best outcome for both players, given the decision of the other player. In particular, notice that from this square in the table that neither player can individually change their decision to become better off; therefore we have a Nash Equilibrium. This tell us these players should always choose this row and column, respectively, when playing this game, which is known as a **Pure Strategy**. This outcome is also the **value** of this game, which is a **$1** payout to the row player. Therefore, if both players were to always get this outcome, then this game is **unfair**.

Problem Set

"*" indicates challenge question

1. Provide your own definition of Game Theory.

2. Explain why the Nash Equilibrium is the best decision for both suspects in the "Prisoner's Dilemma".

3. The game is defined as follows:

   - Two hunters go out to catch meat.
   - There are two rabbits in the range and one stag. The hunters can each bring the equipment necessary to catch one type of animal.
   - The stag has more meat than the rabbits combined, but both hunters must chase the stag to catch it.
   - Rabbit hunters can catch all of their prey by themselves.
   - The values in the table represent the amount of meat (in pounds) the hunters will get depending on what they both choose to hunt.

Hunter 2

Hunter 1		Stag	Rabbit
	Stag	3,3	0,2
	Rabbit	2,0	1,1

Using the Nash Equilibrium, what is the best decision the hunters can make? If there is more than one best decision, explain the pros and cons of each.

4. Complete the table below which represents the game, "Rock, Paper, Scissors". Treat a win as having a value of 1 (in red) for the Row Player, a win as having a value of -1 (in blue) for the Column Player, and treat a draw as having a value of zero. If there is one, find a pure strategy Nash Equilibrium for this game. If not, explain what this could mean when it comes to deciding on a strategy.

Column

Row

	Rock	Paper	Scissors
Rock	0	-1	1
Paper			
Scissors			

5. Look back to the game we played in the "**Best Response - Pure Strategy**" section of the handout. What are the values of these games? Are these games fair?

(a) **Column**

Row

20	0
0	-20

(b)

Column

Row	10	0	-2
	-4	1	-3
	8	-4	-6

(c)

Column

Row	3	-4
	-2	0
	-1	2
	4	3

(d)

Column

Row	3	-1	4	-7
	1	0	2	3
	5	-2	-3	0
	3	0	1	5

6. You are given 10 chocolate coins for you and a friend to share, however you get to decide how many chocolate coins each of you gets (you get 10 and your friend gets 0; you get 9 and your friend gets 1; etc.). After you decide, your friend can either accept the offer or decline it. If they decline it, you both get nothing. How would you split the chocolate coins?

7. * Think of a situation that can arise in your life where you and another person's decision will significantly affect both of you, and create a game to represent this situation.

8. * You are the owner of a clothing store and you must decide at what price to sell a hot new suit. You know that your competitor across the street is selling the same suit, so you must take into consideration the price at which they are selling to make sure you attract as many customers as possible. You have the following information:

    - If you and your competitor both sell the suit at \$50, you will sell 55% of the total number of suits sold between the two of you.
    - If you sell the suit at \$50 but your competitor sells the suit at \$70, you will sell 70% of the total number of suits sold between the two of you.
    - If you sell the suit at \$70 but your competitor sells the suit at \$50, you will sell %40 of the total number of suits sold between the two of you.
    - If you and your competitor both sell the suit at \$70, you will sell %55 of the total number of suit sold between the two of you.

    Represent the above dilemma as a game to answer the following questions:

    (a) If you and your competitor's main goal is to sell as many suits as possible, what is the best price to sell the suit at?

    (b) After doing research, you find out that 100 people will be buying this hot new suit. If you and your competitor's main goal is to maximize the amount of money you make from selling the suit, what is the best price to sell the suit at?

* Mixed Strategy - Expected Value of a Game

Confirm in the game below that there is no pure strategy Nash Equilibrium:

	Column	
Row	4	-3
	-3	5

Because there is no pure strategy for this game, we cannot find the value of this game the way we did before. Instead, we'll assume these players use a mixed strategy, and assign probabilities that represent the likelihood of the row player choosing each respective row and the likelihood of the column player choosing each respective column, then find the **expected** value of the game. Let's do this now:

		Column	
		0.6	0.4
Row	0.4	4	-3
	0.6	-3	5

To find the expected value, E, of this game, we multiply each value in the game by the probability of it occurring, keeping in mind that the players' choices of row or column are independent events, then find the sum of these new values. So, the expected value of this game is:

$$\begin{aligned} E &= (0.6)(0.4)(4) + (0.4)(0.4)(-3) + (0.6)(0.6)(-3) + (0.4)(0.6)(5) \\ &= 0.96 + (\text{-}0.48) + (\text{-}1.08) + 1.2 \\ &= 0.60 \end{aligned}$$

Is this game fair?

9. * Here is our same game of "Rock, Paper, Scissors," except now we are going to use mixed strategies (the numbers in brackets are the probabilities that each player will choose that row or column, respectively):

Column

Row

	Rock(1/3)	Paper(1/3)	Scissors(1/3)
Rock(1/3)	0	-1	1
Paper(1/3)	1	0	-1
Scissors(1/3)	-1	1	0

(a) What is the expected value of this game?

(b) Is this game fair?

(c) Answer questions a and b again for this modified version of the game:

Column

Row

	Rock(1)	Paper(0)	Scissors(0)
Rock(1/3)	0	-1	1
Paper(1/3)	1	0	-1
Scissors(1/3)	-1	1	0

Try multiple probabilities for the column player's decisions, while leaving the row player's mixed strategy at $\frac{1}{3}$ for each decision. What can you say about picking mixed strategies for this game?

10. * The Pirate Game

Five pirates were sailing one day and stumbled upon a treasure chest filled with 10 gold coins. The captain, Nash, and pirates 2, 3, 4, and 5 had to decide how the gold was to be shared. Being the captain, Nash was the first one to make a decision. The rules and conditions of the game are as follows:

Rules and Conditions:

- Nash is to propose how the pirates will share the gold, and the rest of them vote whether or not they agree. If Nash gets at least 50% of the vote in his favour, the gold will be shared his way.
- If the vote is less than 50%, then Nash is thrown off the ship and pirate 2 becomes the captain and the game is repeated.
- The pirates' first goal is to survive and their second goal is to maximize the amount of gold coins they get.
- Assume that all 5 pirates are intelligent, rational, greedy, and do not wish to die, (and are rather good at math for pirates).

Nash finds a plan to maximize his gold and stay alive. What is the plan?