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Grade 6 Math Circles

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Game Theory - Solution Set

1. Provide your own definition of Game Theory.

   Answers may vary. Game theory is the study of two or more people making decisions in a common situation to get the best outcome for themselves, an outcome that is affected by each person's decision.

2. Explain why the Nash Equilibrium is the best decision for both suspects in the "Prisoner's Dilemma".

   Answers may vary. The Nash Equilibrium is the best decision for both suspects in the "Prisoner's Dilemma", because neither suspect could become better off, and leave the other person worse off, by individually changing their decision.

3. The game is defined as follows:

   - Two hunters go out to catch meat.
   - There are two rabbits in the range and one stag. The hunters can each bring the equipment necessary to catch one type of animal.
   - The stag has more meat than the rabbits combined, but both hunters must chase the stag to catch it.
   - Rabbit hunters can catch all of their prey by themselves.
   - The values in the table represent the amount of meat (in pounds) the hunters will get for each given outcome.

		Hunter 2	
		Stag	Rabbit
Hunter 1	Stag	3,3	0,2
	Rabbit	2,0	1,1

Using the Nash Equilibrium of this game, what is the best decision the hunters can make? If there is more than one best decision, explain the pros and cons of each.

First we find the Nash Equilibrium for this game:

		Hunter 2	
		Stag	Rabbit
Hunter 1	Stag	[3,3]	0,2
	Rabbit	2,0	[1,1]

Notice that for both of the outcomes marked above, neither hunter can become better off by individually changing their decision; therefore there are two Nash Equilibriums (Equilibria).

Choosing to hunt the stag is the best decision for both hunters, considering they would maximize the amount of meat they get. However, this decision could be bad, because there is a chance a hunter could get no meat if the other one decides to hunt rabbits.

Choosing to hunt the rabbits is the best decision for both hunters, considering there is no risk of getting no meat. In particular, the worst outcome that can occur is getting 1 pound of meat, and the best outcome is getting 2 pounds of meat. However, this decision is bad, considering both hunters want to maximize the amount of meat they get.

4. Complete the table below which represents the game, "Rock, Paper, Scissors". Treat a win as having a value of 1 for Player 1 (in red) for the Row Player, a win as having a value of -1 (in blue) for the Column Player, and treat a draw as having a value of zero. If there is one, find a pure strategy Nash Equilibrium for this game. If not, explain what this could mean when it comes to deciding on a strategy.

Column

Row

	Rock	Paper	Scissors
Rock	0	[-1, 1]	1, -1
Paper	1, -1	0	[-1, 1]
Scissors	[-1, 1]	1, -1	0

Answers may vary. After performing the "Best Response" method, we see that none of the marked outcomes were the best outcome for both players at the same time. This means that there is no pure strategy for this game; therefore, considering the values for winning are the same for each outcome, the best strategy for both players is to decide what move they are going to make at random.

5. Look back to the game we played in the "**Best Response**" section of the handout. Find the Nash Equilibrium quadrant(s) for each game and comment on the fairness of these games.

Let's perform the "Best Response" method for each of these games.

(a)

	Column	
Row	20	[0]
	0	[-20]

The Nash Equilibrium of this game, or the value of this game, is a payout of $0. Therefore, this game is fair.

(b)

	Column		
	10	0	[-2]
Row	[-4]	1	-3
	8	-4	[-6]

The Nash Equilibrium of this game, or the value of this game, is a payout of $2 to the column player. Therefore, this game is unfair.

(c)

Column

Row	3	[-4]
	[-2]	0
	[-1]	2
	4	[3]

The Nash Equilibrium, or value of this game, is a payout of $3 to the row player. Therefore, this game is unfair.

(d)

Column

Row	3	1	4	[7]
	1	[0]	2	3
	5	2	[3]	0
	3	[0]	1	5

The Nash Equilibriums, or value of this game, are a payouts of $0. Therefore, this game is fair.

6. You are given 10 chocolate coins for you and a friend to share, however you get to decide how many chocolate coins each of you gets (you get 10 and your friend gets 0; you get 9 and your friend gets 1; etc.). After you decide, your friend can either accept the offer or decline it. If they decline it, you both get nothing. How would you split the chocolate coins?

   Answers may vary. There are two cases you must consider:

   (a) Your friend wants as many chocolate coins as possible.

   (b) Your friend wants you to share the coins fairly.

   If the first case is true, then you can give your friend only 1 chocolate coin and keep 9 for yourself, because you know they would rather have 1 chocolate coin than 0.

   If the second case is true, then you should give your friend 5 chocolate coins and keep 5 for yourself, because you know if you do not do this, then you will get 0 chocolate coins.

   This experiment was performed in class. If you would like to see a similar one, follow the link here:
   `http://mindyourdecisions.com/blog/2009/11/03/the-ultimatum-game-played-by-children/#.VFKAIcZqp_Q`

7. * Think of a situation that can arise in your life where you and another person's decision will significantly affect both of you, and create a game to represent this situation.

   Answers may vary.

8. * You are the owner of a clothing store and you must decide at what price to sell a hot new suit. You know that your competitor across the street is selling the same suit, so you must take into consideration the price at which they are selling to make sure you attract as many customers as possible. You have the following information:

- If you and your competitor both sell the suit at \$50, you will sell 55% of the total number of suits sold between the two of you.
- If you sell the suit at \$50 but your competitor sells the suit at \$70, you will sell 70% of the total number of suits sold between the two of you.
- If you sell the suit at \$70 but your competitor sells the suit at \$50, you will sell %40 of the total number of suits sold between the two of you.
- If you and your competitor both sell the suit at \$70, you will sell %55 of the total number of suit sold between the two of you.

Represent the above dilemma as a game to answer the following questions:

(a) If you and your competitor's main goal is to sell as many suits as possible, what is the best price to sell the suit at?

Competitor

		\$50	\$70
You	\$50	55%, 45%	70%, 30%
	\$70	40%, 60%	55%, 45%

Above is the game for the dilemma of what price to sell the suit at. We see the percentage of the total number of suits sold that will be sold by you and your competitor, given the decisions you both make in pricing the suit.

Let's find the Nash Equilibrium of this game using the "Best Response" method.

Competitor

You		$50	$70
	$50	[55%, 45%]	70%, 30%
	$70	[40%, 60%]	55%, 45%

Therefore, if you and your competitor's main goal is to sell as many suits as possible, the best price to sell the suit at is $50 for both of you. Notice that you or your competitor cannot sell more suits if either of you were to individually change the price of the suit.

(b) After doing research, you find out that 100 people will be buying this hot new suit. If you and your competitor's main goal is to maximize the amount of money make from selling the suit, what is the best price to sell the suit at?

The first thing we have to do is find out how much money will be made depending on what price you and your competitor sell the suit as:

- If you both sell the suit at $50:

  You will make: 100 * 0.55 * $50 = $2,750

  Your competitor will make: 100 * 0.45 * $50 = $2250

- If you sell the suit at $50 and your competitor sells it at $70:

  You will make: 100 * 0.70 * $50 = $3500

  Your competitor will make: 100 * 0.30 * $70 = $2100

- If you sell the suit at $70 and your competitor sells it at $50:

You will make: 100 * 0.40 * $70 = $2800

Your competitor will make: 100 * 0.60 * $50 = $3000

- If you both sell the suit at $70:

You will make: 100 * 0.55 * $70 = $3850

Your competitor will make: 100 * 0.45 * $70 = $3150

Now we can make our new game:

Competitor

You		$50	$70
	$50	$2750, $2250	$3500, $2100
	$70	$2800, $3000	$3850, $3150

Let's find the Nash Equilibrium of this game:

Competitor

You		$50	$70
	$50	[2750, 2250]	3500, 2100
	$70	2800, 3000	[3850, 3150]

Therefore, if both you and your competitor's goal is to maximize the amount of money you make from selling the hot new suit, you both should sell the suit at $70.

* Mixed Strategy - Expected Value of a Game

Confirm in the game below that there is no pure strategy Nash Equilibrium:

Column

Row

4	-3
-3	5

Because there is no pure strategy for this game, we cannot find the value of this game the way we did before. Instead, we'll assume these players use a mixed strategy, and assign probabilities that represent the likelihood of the row player choosing each respective row and the likelihood of the column player choosing each respective column, then find the **expected** value of the game. Let's do this now:

Column

Row

	0.6	0.4
0.4	4	-3
0.6	-3	5

To find the expected value, E, of this game, we multiply each value in the game by its probability of it occurring, keeping in mind that the players' choices of row or column are independent events, then find the sum of these new values. So, the expected value of this game is: $E = (0.6)(0.4)(4) + (0.4)(0.4)(-3) + (0.6)(0.6)(-3) + (0.4)(0.6)(5)$

$$= 0.96 + (-0.48) + (-1.08) + 1.2$$

$$= 0.60$$

Is this game fair? No, because we expect the value of this game to be a payout of \$0.60 to the row player.

9. * Here is our same game of "Rock, Paper, Scissors," except now we are going to use mixed strategies (the numbers in brackets are the probabilities that each player will choose that row or column, respectively):

Column

Row		Rock(1/3)	Paper(1/3)	Scissors(1/3)
	Rock(1/3)	0	-1	1
	Paper(1/3)	1	0	-1
	Scissors(1/3)	-1	1	0

(a) What is the expected value of this game?

Again, we multiply each value in this game by its probability of occurring, then find the sum of these new values. So the expected value of this game is:

$$
\begin{aligned}
E &= (\frac{1}{3})(\frac{1}{3})(0) + (\frac{1}{3})(\frac{1}{3})(-1) + (\frac{1}{3})(\frac{1}{3})(1) + (\frac{1}{3})(\frac{1}{3})(1) + (\frac{1}{3})(\frac{1}{3})(0) \\
&\quad + (\frac{1}{3})(\frac{1}{3})(-1) + (\frac{1}{3})(\frac{1}{3})(-1) + (\frac{1}{3})(\frac{1}{3})(1) + (\frac{1}{3})(\frac{1}{3})(0) \\
&= 0 - (\frac{1}{9}) + \frac{1}{9} + \frac{1}{9} + 0 - (\frac{1}{9}) - (\frac{1}{9}) + \frac{1}{9} + 0 \\
&= -(\frac{1}{9}) + \frac{1}{9} + \frac{1}{9} - (\frac{1}{9}) - (\frac{1}{9}) + \frac{1}{9} \\
&= 0 + 0 + 0 \\
&= 0
\end{aligned}
$$

(b) Is this game fair?

Since we expect the value of this game to be 0, this game is fair.

(c) Answer questions a and b again for this modified version of the game:

Column

Row

	Rock(1)	Paper(0)	Scissors(0)
Rock(1/3)	0	-1	1
Paper(1/3)	1	0	-1
Scissors(1/3)	-1	1	0

Since the column player will play rock 100% of the time, when finding the expected value of this game we only need to consider the case that the column player plays rock:

$$\begin{aligned} E &= (1)(\frac{1}{3})(0) + (1)(\frac{1}{3})(1) + (1)(\frac{1}{3})(-1) \\ &= 0 + \frac{1}{3} - \frac{1}{3} \\ &= 0 \end{aligned}$$

Since we expect the value of this game to be 0, this is a fair game.

Try multiple probabilities for the column player's decisions, while leaving the row player's mixed strategy at $\frac{1}{3}$ for each decision. What can you say about picking mixed strategies for this game?

As long as the row player keeps their mixed strategy at $\frac{1}{3}$ for all of their decisions, then it does not matter what the column player chooses as his mixed strategy, because the expected value of this game will always be 0. Therefore, because we cannot know what the other player's strategy is, it is safest for a player to use a mixed strategy of $\frac{1}{3}$ for their choices of rock, paper, or scissors when playing this game.

10. * The Pirate Game

Five pirates were sailing one day and stumbled upon a treasure chest filled with 10 gold coins. The captain, Nash, and pirates 2, 3, 4, and 5 had to decide how the gold was to be shared. Being the captain, Nash was the first one to make a decision. The rules and conditions of the game are as follows:

Rules and Conditions:

- Nash is to propose how the pirates will share the gold, and the rest of them vote whether or not they agree. If Nash gets at least 50% of the vote in his favour, the gold will be shared his way.
- If the vote is less than 50%, then Nash is thrown off the ship and pirate 2 becomes the captain and the game is repeated.
- The pirate making the offer also has a vote.
- The pirates' first goal is to survive and their second goal is to maximize the amount of gold coins they get.
- Assume that all 5 pirates are intelligent, rational (will accept a low amount of gold if they have to), greedy, and do not wish to die, (and are rather good at math for pirates).

Nash finds a plan to maximize his gold and stay alive. What is the plan?

Answers may vary, but there is the optimal solution:

The way we find the optimal solution is by working backwards to see if we can find any situations where a pirate will accept a low amount of gold. Remember, Nash wants to give away the least amount of gold coins as possible. If we can find these situations, then Nash, being an intelligent pirate, will also be able to find them.

Assume that pirates Nash, 2, and 3 have been thrown off the ship after their offers of how to share the gold were rejected. We are then left with pirates 4 and 5:

4	10
5	0

In this situation, pirate 4 can take all 10 gold coins for himself and still get 50% of the votes, since his vote counts. Therefore, pirate 5, being greedy, must do what is necessary to avoid this situation where its just him and pirate 4 on the ship. So, now, let's assume pirate 3 is back on the ship:

3	9
4	0
5	1

In this situation, pirate 3 can offer pirate 5 only 1 gold coin, because, being intelligent, he knows that if pirate 5 does not accept this offer then pirate 5 will get 0 gold coins when it is pirate 4's turn to make an offer. So pirate 5 will accept this offer, being rational, and pirate 3 will get at least 50% of the vote. However, pirate 4, being greedy, looks at this situation and realizes that he must do what is necessary to avoid it, or else he will end up with zero gold coins. So, now, let's assume pirate 2 is back on the ship:

2	9
3	0
4	1
5	0

In this situation, pirate 2 can offer pirate pirate 4 only 1 gold coin, because, being intelligent, he knows that if pirate 4 does not accept this offer then pirate 4 will get 0 gold coins when it is pirate 3's turn to make an offer. So pirate 4 will accept this offer, being rational, and pirate 2 will get at least 50% of the vote. However, pirates 3 and 5 look at this situation and realize that they must do what is necessary to avoid it, or else they will end up with zero gold coins. So, now, let's assume pirate Nash is back on the ship:

Nash	8
2	0
3	1
4	0
5	1

In this situation, pirate Nash can offer pirates 3 and 5 each one gold coin, because, being intelligent, he knows that if pirates 3 and 5 do not accept this offer then they will both get 0 gold coins when it is pirate 2's turn to make an offer. So pirates 3 and 5 will accept this offer, being rational, and pirate Nash will get at least 50% of the vote. Therefore, Nash should offer pirates 3 and 5 one gold coin each and keep eight for himself.