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Grade 7/8 Math Circles

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How We Make Money

Finding a Job

Today we will talk about ways you can go out and make money in today's economy. The first and most obvious way is going out and finding a paying job. One type of paying job you can find is selling valuable items, or services, and earning commission.

Commission

Commission is when you sell something for a certain amount of x dollars and take a portion of the money made for yourself. This portion is represented by a percentage called the commission rate. Employers offer pay their employees with commission to motivate them to sell more.

Problem 1: You found a job selling boxes of chocolate almonds that pays a commission rate of 20%. The price of a box of chocolate almonds is \$4.

1. How much commission do you make selling one box of chocolate almonds?

$$\text{Commission} = 20\% \text{ of } \$4 = 0.20(\$4) = \$0.80$$

$$\text{or since } 0.2 = \frac{1}{5}$$

$$\text{Commission} = 20\% \text{ of } \$4 = \frac{\$4}{5} = \$0.80$$

Therefore, you make \$0.80 for every box of chocolate almonds you sell.

2. If you sell \$100 worth of chocolate almonds, how much commission do you make?

$$\text{Commission} = 0.20(\$100) = \$20$$

or

$$\text{Commission} = \frac{\$100}{5} = \$20$$

Therefore, you make \$20 of commission.

3. If you need to make \$200, how many boxes of chocolate almonds must you sell?

Let b be the amount of boxes of chocolate almonds you must sell to make \$200. Then

$$\begin{aligned} 0.80b &= 200 \\ \frac{4}{5}b &= 200 \\ 5 * \frac{4}{5}b &= 5 * 200 \\ 4b &= 1000 \\ \frac{4b}{4} &= \frac{1000}{4} \\ b &= 250 \end{aligned}$$

Therefore, you must sell 250 boxes of chocolate almonds to make \$200.

Commission With an Hourly Wage

Sometimes, when it's your job to sell something you are paid an additional hourly wage on top of the commission you make. This is just in case you are not able to sell anything and still need to make money.

Problem 2: You found a new job working at a sports store. The job pays a commission rate of 5% for anything you sell in the store, and an hourly wage of \$12 per hour. On your first day you started at 10:00a.m., left for lunch at 1:00p.m., and came back at 2:00pm.

1. In the morning, you sold a pair of goalie pads for \$400, two hockey sticks for \$250 each, and a football for \$20. How much money did you earn before you went to lunch?

You worked from 10:00a.m. to 1:00p.m. and so earned $3 * \$12 = \36 with your hourly wage.

Money earned with commission

$$= (400 + 2 * 250 + 20)(0.05) = (920)(0.05) = 46$$

$$\text{commission} + \text{hourly wage} = \$46 + \$36 = \$82$$

Therefore, you made \$82 before you went to lunch.

2. After lunch, you worked for 4 more hours and sold a basketball net for \$300 and two tennis rackets for \$60 each. What is the total amount of money you have earned so far?

Combining the hourly wage and commission in one step we have

$$\begin{aligned} & 82 + 4(12) + (300 + 120)(0.05) \\ &= 82 + 48 + 420(0.05) \\ &= 130 + 21 \\ &= 151 \end{aligned}$$

Therefore, the total amount you have earned so far is \$151.

2. Your shift is ending at 7:00p.m. and you have one more customer who has scheduled an appointment within that time to purchase a pair of roller skates. If your goal is to have made \$170 at the end of your shift, what price must you sell the roller skates at?

You came back from lunch at 2:00p.m. and worked 4 more hours, which means it is now 6:00p.m. You only have 1 more hour to work.

Let x be the price of the roller skates in dollars. Then

$$\begin{aligned}
 0.05x + 1(12) + 151 &= 170 \\
 0.05x &= 170 - 151 - 12 \\
 0.05x &= 7 \\
 \frac{1}{20}x &= 7 \\
 20 * \frac{1}{20}x &= 7 * 20 \\
 20 * \frac{1}{20}x &= 140 \\
 x &= 140
 \end{aligned}$$

Therefore, you must sell the roller skates for \$140.

Graduated Commission

Another way your employers encourage you to sell more is by offering **graduated commission**—once you sell a certain amount of items, or services, the commission rate increases for every sale after that, giving you more commission per sale.

Problem 3: You find a job making Rubik's Cubes at Rubik's Factory. You get paid \$13 an hour and \$1 for the first 15 cubes you make, \$2 for the next 10, and \$3 for any cubes after that. It takes you 15 minutes to make one Rubik's Cube.

- How much money do you make if you work for 8 hours at Rubik's Factory?

Firstly, you make 4 cubes in an hour since it takes you 15 minutes to make 1 ($\frac{60}{15} = 4$). This means that you make $4 * 8 = 32$ cubes in 8 hours of work at the factory. So, for how much you make in 8 hours:

$$\begin{aligned} & 8(13) + 1(15) + 2(10) + 3(7) \\ &= 104 + 15 + 20 + 21 \\ &= 160 \end{aligned}$$

Therefore, you make \$160 if you work for 8 hours at Rubik's Factory.

- After the first week, Rubik changes your shift to only 5 hours. However, you want to earn the same amount of money as you did working for 8 hours. How many cubes must you make per hour?

Firstly, let's find out how many cubes you must make in 5 hours. You know that you are going to have to make more cubes than you did working the 8 hour shift, because you are now earning less with your hourly wage ($5 * 13 = 65$ vs. $8 * 13 = 106$). So, you just need to find out how many cubes you must make when earning the \$3 commission. You make 15 cubes at the \$1 commission, 10 cubes at the \$2 commission, and c cubes at the \$3 commission. Let's solve for c , knowing that you want to make \$160.

$$\begin{aligned} 5(13) + 1(15) + 2(10) + 3c &= 160 \\ 65 + 15 + 20 + 3c &= 160 \\ 100 + 3c &= 160 \\ 3c &= 160 - 100 \\ 3c &= 60 \\ \frac{3}{3}c &= \frac{60}{3} \\ c &= 20 \end{aligned}$$

Therefore, you must make $10 + 15 + 20 = 45$ Rubik's Cubes in 5 hours, or 9 Rubik's Cubes per hour ($\frac{45\text{cubes}}{9\text{hours}}$).

3. Rubik changes your shift back to 8 hours, except now he pays you the same commission rate regardless of how many cubes you make. If he promises to pay you \$225 for the whole shift and you make 9 cubes per hour, what is the new commission rate?

In 8 hours you make $8 * 9 = 72$ cubes. Let r be the new commission rate that Rubik pays you. Then

$$\begin{aligned}
 8(13) + r(72) &= 225 \\
 104 + 72r &= 225 \\
 72r &= 225 - 104 \\
 72r &= 121 \\
 \frac{72}{72}r &= \frac{121}{72} \\
 \cancel{72}^r &= \frac{121}{72} \\
 r &= \frac{121}{72} \\
 &\approx 1.68
 \end{aligned}$$

Therefore, the new commission rate is about \$1.68 per cube.

Earning Interest

Once you have found a job and earned some money, you can then invest that money to earn even more by receiving interest. What's meant by "investing and receiving interest", is just lending someone a certain amount of money who then pays you back that same amount plus a little extra (the interest) for the favour you did them. The interest will always be a percentage of the amount of money you invested, a percentage called the **interest rate**. The first type of interest we'll discuss is **Simple Interest**.

Simple Interest

Under simple interest, the interest earned is calculated by multiplying the original amount invested, called the principal, by the yearly interest rate, which is multiplied by the length of the investment in years.

The formula to calculate simple interest, I , is

$$\boxed{I = Prt}$$

Where P is the principal amount (or original amount) invested, r is the yearly interest rate, and t is the time in years.

Example: You invest \$100 at the local bank that pays simple interest at the end of the year at an interest rate of 5%.

- How much interest have you earned at the end of the year? How much money do you have now?

$$I = ?$$

$$P = \$100$$

$$r = 5\%$$

$$t = 1$$

$$I = Prt$$

$$= \$100(0.05)(1)$$

$$= \$5$$

Therefore, you earn \$5 at the end of one year and now have $\$100 + \$5 = \$105$. This \$105 is called the **accumulated value**. For this example, accumulated value just means the future value of \$100 after one year with simple interest of 5%

Accumulated Value

The accumulated value of an investment, A , is the amount of money you invested plus the interest you earned. So the accumulated value of an investment under simple interest is

$$\begin{aligned} A &= P + I \\ &= P + Prt \\ &= P(1 + rt) \end{aligned}$$

Where, again, P is the principal amount invested, r is the yearly interest rate, and t is the time in years.

Problem 4: You earned \$200 in October and now want to invest your money. You find an owner of a lemonade stand in need of some lemons, so offer to buy them \$200 worth of lemons.

1. If you charge simple interest at a yearly rate of 10%, how much money will you have accumulated in 6 months?

$$\begin{array}{ll} A = ? & A = P(1 + rt) \\ P = \$200 & = \$200[1 + 0.10(\frac{1}{2})] \\ r = 10\% & = \$200[1 + 0.05] \\ t = \frac{1}{2} & = \$210 \end{array}$$

Therefore, in 6 months you will have accumulated \$210.

2. The owner of the lemonade stand says that \$200 worth of lemons is too much. Instead, they tell you that they will pay you back \$200 in total at the end of one year at a yearly interest rate of 15%. How many dollars worth of lemons must you buy for the owner of the lemonade stand?

$$\begin{aligned}
 P(1 + rt) &= A \\
 P &=? & P[1 + 0.15(1)] &= 200 \\
 A &= 200 & P[1 + 0.15] &= 200 \\
 r &= 15\% & P &= \frac{200}{[1 + 0.15]} \\
 t &= 1 & &= \frac{200}{[1.15]} \\
 & & &\approx 173.91
 \end{aligned}$$

Therefore, you must buy about \$173.91 worth of lemons for the owner of the lemonade stand. This value of \$173.91 is known as the **present value** of \$200 at a simple interest rate of 15% over one year. The formula for the present value is: $P = \frac{A}{(1 + rt)}$.

3. How long would it take, in years, for the owner of lemonade stand to pay you back \$180 if you bought them \$150 worth of lemons at an interest rate of 10%?

$$\begin{aligned}
 P(1 + rt) &= A \\
 150[1 + 0.10t] &= 180 \\
 \frac{150}{150}[1 + 0.10t] &= \frac{180}{150} \\
 \frac{150}{150}[1 + 0.10t] &= \frac{6}{5} \\
 t &=? & 1 + 0.10t &= \frac{6}{5} \\
 P &= 150 & 0.10t &= \frac{6}{5} - 1 \\
 A &= 180 & 0.10t &= \frac{6}{5} - \frac{5}{5} \\
 r &= 10\% & \frac{1}{10}t &= \frac{1}{5} \\
 & & 10 * \frac{1}{10}t &= 10 * \frac{1}{5} \\
 & & 10 * \frac{1}{10}t &= \frac{10}{5} \\
 & & t &= 2
 \end{aligned}$$

Therefore, it would take 2 years for the owner to pay you back \$180 in total.

Compound Interest

Under compound interest, the interest earned is calculated by multiplying the principal amount and the interest already earned by the yearly interest rate.

Example: Assume that you have deposited \$1000 into a savings account that pays 10% compounded annually

After one year, the money earns $0.10 * \$1000 = \100 . The new balance is then \$1100.

After two years, the money earns $0.10 * \$1100 = \110 . The new balance is then \$1210.

After three years, the money earns $0.10 * \$1210 = \121 . The new balance is then \$1331.

And so on.

Under compound interest, the formula to calculate the accumulated value, A , is

$$A = P(1 + r)^t$$

Where P is the principal amount, r is the annually compounded interest rate, and t is the time in years.

Problem: You open up a savings account that pays 4% compounded annually.

1. If you deposit \$100, how much money will you save in 3 years?

$$A = ?$$

$$P = 100$$

$$r = 0.04$$

$$t = 3$$

$$A = 1000(1 + 0.04)^3$$

$$= 1000(1.04)^3$$

$$= 1124.86$$

Therefore, you will save \$1124.86 in 3 years.

2. If in 6 years you will have saved \$2530.64, how much money did you deposit?

$$\begin{array}{ll}
 P = ? & P(1 + 0.04)^6 = 2530.64 \\
 A = 2530.64 & P \frac{(1.04)^6}{(1.04)^6} = \frac{2530.64}{(1.04)^6} \\
 r = 0.04 & P \frac{(1.04)^6}{(1.04)^6} = 2000 \\
 t = 6 & P = 2000
 \end{array}$$

Therefore, you deposited \$2000.

3. What would the interest rate be if you saved \$1650 with an initial deposit of \$1500 two years ago.

$$\begin{array}{ll}
 r = ? & 1500(1 + r)^2 = 1650 \\
 P = 1500 & \frac{1500}{1500}(1 + r)^2 = \frac{1650}{1500} \\
 A = 1650 & \frac{1500}{1500}(1 + r)^2 = 1.10 \\
 t = 2 & (1 + r)^2 = 1.10 \\
 & \sqrt{(1 + r)^2} = \sqrt{1.10} \\
 & 1 + r = 1.0488 \\
 & r = 4.88
 \end{array}$$

Therefore, the interest would be 4.88% compounded annually.

Problem Set

“*” indicates challenge question

1. You find a job selling water that pays a 10% commission rate on the sales you make. If you sold \$150 worth of water in a week, how much commission did you make that week?

$$\text{Commission earned} = 10\% \text{ of } \$150 = 0.10(\$150) = \$15$$

Therefore, you made \$15 of commission that week.

2. You invest \$50 into a savings account at the local bank that pays simple interest at a interest rate of 5% per year. How much interest have you earned in 10 months?

$$I = ?$$

$$P = \$50$$

$$r = 5\%$$

$$t = \frac{10}{12}$$

$$I = Prt$$

$$= 50(0.05)\left(\frac{10}{12}\right)$$

$$= 12.50$$

Therefore, in 10 months you have earned \$12.50 of interest. *Note: Since there are 12 months in a year, the time is $t = \frac{10}{12}$ years.

3. You find a job painting garages that pays an hourly wage of \$12 per hour and a commission rate of 7%. If the company you work for charges \$50 for every garage they paint, how much money did you make your first day if you painted 3 garages in 5 hours?

$$\text{Money made with hourly wage} = 5(\$12) = \$60$$

$$\text{Money made with commission} = 3(\$50)(0.07) = \$10.50$$

$$\text{Commission + hourly wage} = \$10.50 + \$60 = \$70.50$$

Therefore, you made \$70.50 your first day.

4. You lend \$500 to a start-up clothing company called “RVS” and demand that they pay you back in 2 years at an interest rate of 20% per year compounded annually. How much will “RVS” pay you in full at the end of 2 years?

$$A = ?$$

$$P = 500$$

$$r = 20\%$$

$$t = 2$$

$$A = P(1 + r)^t$$

$$A = 500(1 + 0.20)^2$$

$$= 500(1.20)^2$$

$$= 720$$

Therefore, they will pay you back \$720 in 2 years.

5. You find a job cutting hair that pays you with graduated commission. You make \$10 per haircut for your first 2 haircuts, \$12 per haircut for your next 3, and \$14 per haircut for any amount after that in any given day of work. How many haircuts must you provide to make \$126 in one day?

Thinking logically, if you provide 5 haircuts, you will make

$$2(\$10) + 3(\$12) = \$20 + \$36 = \$56$$

which is short of your goal of \$126. Therefore, you only have to find out how many haircuts you must provide at the \$14 commission rate, since you know how many haircuts you must provide at the \$10 and \$12 commission rates (2 haircuts and 3 haircuts, respectively). Call this amount of haircuts at the \$14 commission rate x . Then

$$\begin{aligned} 2(10) + 3(12) + x(14) &= 126 \\ 56 + 14x &= 126 \\ 14x &= 126 - 56 \\ 14x &= 70 \\ \frac{14}{14}x &= \frac{70}{14} \\ x &= 5 \end{aligned}$$

Therefore, you must provide $2 + 3 + 5 = 10$ haircuts to make \$126 in one day.

6. From the problem above- your employer takes away the graduated commission and pays you with one commission rate regardless of how many haircuts you provide. If he promises to still pay you \$126 for the same amount of haircuts, what is the new commission rate?

You will make the same amount of money per haircut for 10 haircuts. So, for the commission rate, c , this means that

$$\begin{aligned} 10c &= 126 \\ \frac{10}{10}c &= \frac{126}{10} \\ c &= 12.6 \end{aligned}$$

Therefore, the new commission rate is \$12.6 per haircut.

7. A friend that you lent money to one month ago has paid you back \$60 in full. If you originally charged them with simple interest at an interest rate of 84% per year, what was the original amount you lent them?

$$\begin{array}{ll}
 P = ? & P(1 + rt) = A \\
 A = 60 & P[1 + 0.84(\frac{1}{12})] = 60 \\
 r = 84\% & P[1 + 0.07] = 60 \\
 t = \frac{1}{12} & P = \frac{60}{[1 + 0.07]} \\
 & P \approx 56.07
 \end{array}$$

Therefore, the original amount you lent your friend was \$56.07.

8. You see an advertisement in the newspaper that two different employers are hiring for the same job of selling skis. One employer pays 15\$ per hour and the other pays 5% commission for your sales. How many dollars worth of skis must you sell per hour so that both jobs are just as good?

Let x be the amount of dollars worth of skis you must sell in an hour. So you need

$$0.05x = 15 \rightarrow \frac{1}{20}x = 15 \rightarrow x = 15 * 20 \rightarrow x = 300$$

Therefore, you must sell \$300 worth of skis for both jobs to be just as good.

9. Prove that for any investment, $\$P$, that pays an annual interest rate, r , and lasts exactly 1 year that the accumulated value is the same under simple and compound interest.

Under compound interest: $A = P(1 + r)^t = P(1 + r)^1 = P(1 + r)$, for $t = 1$.

Under simple interest: $A = P(1 + rt) = P(1 + t(1)) = P(1 + r)$, for $t = 1$.

10. * You are trying to decide which student you should teach to skate. Kayla pays you \$30 to show up and \$15 per hour for each lesson. Raheem pays you \$20 per hour for each lesson. How many hours do both lessons need to be so that teaching both students is just as good?

Let t be the amount of hours both lessons.

For Kayla you make: $15t + 30$ dollars for a t hour lesson

For Raheem you make: $20t$ dollars for a t hour lesson

We need to make the same amount of dollars teaching either student so we make the equation:

$$\begin{aligned} 20t &= 15t + 30 \\ 20t - 15t &= 30 \\ 5t &= 30 \\ \frac{5}{5}t &= \frac{30}{5} \\ t &= 6 \end{aligned}$$

Therefore, both lessons need to be 6 hours so that teaching both students is just as good.

11. * You wish to accumulate \$250 with an investment of \$200 over 3 years. What interest rate compounded annually must you ask for? Round to the nearest thousandth.

$$\begin{aligned} r &=? \\ P &= 200 \\ A &= 250 \\ t &= 3 \\ 200(1+r)^3 &= 250 \\ \frac{200}{200}(1+r)^3 &= \frac{250}{200} \\ \frac{200}{200}(1+r)^3 &= 1.25 \\ (1+r)^3 &= 1.25 \\ \sqrt[3]{(1+r)^3} &= \sqrt[3]{1.25} \\ 1+r &= 1.0772 \\ r &= 7.72\% \end{aligned}$$

Therefore, you must ask for an interest rate of 7.72% compounded annually.

12. * There are two investment options:

- (a) Invest \$100 at an interest rate r compounded annually over 4 years.
- (b) Invest \$50 at an interest rate r compounded annually over 8 years.

Solve for the interest rate r compounded annually such that these two investment options accumulate to the same value. Round to the nearest thousandth.

For the first investment option we have: $A = 100(1 + r)^4$

For the second investment option we have: $A = 50(1 + r)^8$

These two investment options have to accumulate to the same value, so we have:

$$\begin{aligned}
 100(1 + r)^4 &= 50(1 + r)^8 \\
 \frac{100}{100}(1 + r)^4 &= \frac{50}{100}(1 + r)^8 \\
 (1 + r)^4 &= \frac{1}{2}(1 + r)^8 \\
 \frac{(1 + r)^4}{(1 + r)^4} &= \frac{\frac{1}{2}(1 + r)^8}{(1 + r)^4} \\
 1 &= \frac{1}{2}(1 + r)^4 \\
 2 &= (1 + r)^4 \\
 \sqrt[4]{2} &= \sqrt[4]{(1 + r)} \\
 1.1892 &= 1 + r \\
 18.9\% &= r
 \end{aligned}$$

Therefore, an interest rate of 18.92% compounded annually will make these two investment options accumulate to the same value.

13. * One summer you worked as a sales person selling laptops. The job paid you \$22 per hour and a commission rate c . You invested all of the money you made working that summer into a savings account that offered an interest rate of 10% compounded annually. After 3 years your savings account has accumulated \$13310. If you worked 40 hours per week for 8 weeks and sold \$11840 worth of laptops that summer, 3 years ago, what was the commission rate of your job selling laptops.

Let c be the commission rate of your job selling laptops. Then

$$\text{Money earned that summer} = 8(40)(22) + 11840c = 7040 + 11840c$$

But, we know that

$$\text{Money earned that summer} = \frac{13310}{(1 + 0.10)^3} = 10000$$

This is because we invested the money we earned and it accumulated to 13310 in 3 years at an interest rate of 10% compounded annually.

So then

$$\text{Money earned that summer} = 7040 + 11840c$$

Becomes

$$10000 = 7040 + 11840c$$

Now we just need to solve for c :

$$\begin{aligned} 11840c + 7040 &= 10000 \\ 11840c &= 10000 - 7040 \\ 11840c &= 2960 \\ \frac{11840}{11840}c &= \frac{2960}{11840} \\ c &= 0.25 \end{aligned}$$

Therefore, the commission rate of your job selling laptops was 25%.

14. * You lent a certain amount of money to a start-up chocolate bar company. To repay you, the company pays you \$10 after the first year, \$20 dollars after the second year, and repeats this payment process of paying 10 more dollars each year for 50 years (10, 20, 30, 40...). They tell you that this will be the total amount of interest you earn for lending them money. If the yearly simple interest rate was 5% when you made the agreement to lend them money, how much money did you lend them.

Your interest earned is

$$I = 10 + 20 + 30 + 40 + \dots + 500$$

Which is the same as

$$I = 10(1 + 2 + 3 + 4 + \dots + 50)$$

But, the terms in the brackets form an arithmetic series that increases by 1 with each term.

Using our knowledge of series we have

$$I = 10\left(\frac{51 * 50}{2}\right) = 12750$$

Now, using our formula for the amount of interest earned under simple interest we have

$$\begin{aligned} I &= Prt \\ 12750 &= P(0.05)(50) \\ 12750 &= P(2.5) \\ \frac{12750}{2.5} &= P \\ P &= 5100 \end{aligned}$$

Therefore, you lent them \$5100.