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Grade 7/8 Math Circles

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Game Theory - Solution Set

1. Look back to the game we played in the "**Example - Find the Nash Equilibrium**" section of the handout. What are the Nash Equilibriums of these games? Are these games fair?

(a) **Column**

Row

20	[0]
0	[20]

The Nash Equilibrium of this game is a payout of $0. Therefore, this game is fair.

(b) **Column**

Row

10	0	[2]
[4]	1	3
8	4	[6]

The Nash Equilibrium of this game is a payout of $2 to the column player. Therefore, this game is unfair.

(c)

Column

Row

3	[4]
[2]	0
[1]	2
4	[3]

The Nash Equilibrium is a payout of $3 to the row player. Therefore, this game is unfair.

(d)

Column

Row

3	1	4	[7]
1	[0]	2	3
5	2	[3]	0
3	[0]	1	5

The Nash Equilibriums are a payouts of $0. Therefore, this game is fair.

2. Complete the table below which represents the game, "Rock, Paper, Scissors". Treat a win as having a value of 1 for Player 1 (in red) for the Row Player, a win as having a value of 1 (in blue) for the Column Player, and treat a draw as having a value of zero. If there is one, find a best strategy for this game. If not, explain what this could mean when it comes to deciding on a strategy.

Column

Row

	Rock	Paper	Scissors
Rock	0	[1]	(1)
Paper	(1)	0	[1]
Scissors	[1]	(1)	0

Answers may vary. Notice we could not find a *best strategy* for this game; therefore, considering the values for winning are the same for each outcome, the best strategy for both players is to decide what move they are going to make at random.

3. Alyssa and Tom are making treats for a bake sale to raise money for new calculators for the school. They are having a hard time deciding whether to make cookies or brownies, since person who sells the most treats will win a prize. After doing some research within the school, they both find out the following information:

   - If Tom and Alyssa both make cookies, then Tom will sell 35 cookies and Alyssa will sell 45 cookies.
   - If Tom and Alyssa both make brownies, then Tom will sell 20 brownies and Alyssa will sell 30 brownies.
   - If Tom makes cookies and Alyssa makes brownies, then Tom will sell 60 cookies and Alyssa will sell 40 brownies.
   - If Tom makes brownies and Alyssa makes cookies, then Tom will sell 30 brownies and Alyssa will sell 70 cookies.

Alyssa

Tom

	Cookies	Brownies
Cookies	35,45	60,40
Brownies	30,70	20,30

Using the information above to fill in the table with how many treats they each will sell depending on what treat they decide to make. Then

(a) Find the Nash Equilibrium of this game to find out what type of treat each student will make.

(b) Is this a fair competition?

(c) If not, what can Tom do to ensure the bake sale is as successful as possible, regardless if he wins or not?

(a)

Alyssa

Tom		Cookies	Brownies
	Cookies	[35,45]	60,40
	Brownies	30,70	20,30

The Nash Equilibrium of this game tells that that both students will make cookies.

(b) Since the best strategy for both students does not lead to an equal outcome, this competition is not fair.

(c) Let's make another game table to see the combined amount of treats both students will sell together.

Alyssa

Tom		Cookies	Brownies
	Cookies	80	100
	Brownies	100	50

Therefore, Tom should propose to Alyssa that one of them makes cookies and the other makes brownies.

4. The game is defined as follows:

    - Two hunters go out to catch meat.
    - There are two rabbits in the range and one stag. The hunters can each bring the equipment necessary to catch one type of animal.
    - The stag has more meat than the rabbits combined, but both hunters must chase the stag to catch it.
    - Rabbit hunters can catch all of their prey by themselves.
    - The values in the table represent the amount of meat (in pounds) the hunters will get for each given outcome.

Hunter 2

Hunter 1		Stag	Rabbit
	Stag	3,3	0,2
	Rabbit	2,0	1,1

Using the Nash Equilibrium of this game, what is the best decision the hunters can make? If there is more than one best decision, explain the pros and cons of each.

First we find the Nash Equilibrium for this game:

Hunter 2

Hunter 1

	Stag	Rabbit
Stag	[(3,3)]	0,2
Rabbit	2,0	[(1,1)]

Notice that for both of the outcomes marked above, neither hunter can become better off by individually changing their decision; therefore there are two Nash Equilibriums (Equilibria).

Choosing to hunt the stag is the best decision for both hunters, considering they would maximize the amount of meat they get. However, this decision could be bad, because there is a chance a hunter could get no meat if the other one decides to hunt rabbits.

Choosing to hunt the rabbits is the best decision for both hunters, considering there is no risk of getting no meat. In particular, the worst outcome that can occur is getting 1 pound of meat, and the best outcome is getting 2 pounds of meat. However, this decision is bad, considering both hunters want to maximize the amount of meat they get.

5. * You are the owner of a clothing store and you must decide at what price to sell a hot new suit. You know that your competitor across the street is selling the same suit, so you must take into consideration the price at which they are selling to make sure you attract as many customers as possible. You have the following information:

- If you and your competitor both sell the suit at \$50, you will sell 55% of the total number of suits sold between the two of you.
- If you sell the suit at \$50 but your competitor sells the suit at \$70, you will sell 70% of the total number of suits sold between the two of you.
- If you sell the suit at \$70 but your competitor sells the suit at \$50, you will sell %40 of the total number of suits sold between the two of you.
- If you and your competitor both sell the suit at \$70, you will sell %55 of the total number of suit sold between the two of you.

Represent the above information as a game to answer the following questions:

(a) If you and your competitor's main goal is to sell as many suits as possible, what is the best price to sell the suit at?

Competitor

You		\$50	\$70
	\$50	55%, 45%	70%, 30%
	\$70	40%, 60%	55%, 45%

Above is the game for what price to sell the suit at. We see the percentage of the total number of suits sold that will be sold by you and your competitor, given the decisions you both make in pricing the suit.

Let's find the Nash Equilibrium of this game using the "Best Response" method.

Competitor

		$50	$70
You	$50	[(55%, 45%)]	(70%, 30%)
	$70	[40%, 60%]	55%, 45%

Therefore, if you and your competitor's main goal is to sell as many suits as possible, the best price to sell the suit at is $50 for both of you. Notice that you or your competitor cannot sell more suits if either of you were to individually change the price of the suit.

(b) After doing research, you find out that 100 people will be buying this hot new suit. If you and your competitor's main goal is to maximize the amount of money make from selling the suit, what is the best price to sell the suit at?

The first thing we have to do is find out how much money will be made depending on what price you and your competitor sell the suit as:

- If you both sell the suit at $50:

  You will make: 100 * 0.55 * $50 = $2,750

  Your competitor will make: 100 * 0.45 * $50 = $2250

- If you sell the suit at $50 and your competitor sells it at $70:

  You will make: 100 * 0.70 * $50 = $3500

  Your competitor will make: 100 * 0.30 * $70 = $2100

- If you sell the suit at \$70 and your competitor sells it at \$50:

  You will make: 100 * 0.40 * \$70 = \$2800

  Your competitor will make: 100 * 0.60 * \$50 = \$3000

- If you both sell the suit at \$70:

  You will make: 100 * 0.55 * \$70 = \$3850

  Your competitor will make: 100 * 0.45 * \$70 = \$3150

Now we can make our new game:

Competitor

You		\$50	\$70
	\$50	\$2750, \$2250	\$3500, \$2100
	\$70	\$2800, \$3000	\$3850, \$3150

Let's find the Nash Equilibrium of this game:

Competitor

You		\$50	\$70
	\$50	[2750, 2250]	3500, 2100
	\$70	(2800, 3000)	[(3850, 3150)]

Therefore, if both you and your competitor's goal is to maximize the amount of money you make from selling the hot new suit, you both should sell the suit at \$70.

6. You are given 10 chocolate coins for you and a friend to share, however you get to decide how many chocolate coins each of you gets (you get 10 and your friend gets 0; you get 9 and your friend gets 1; etc.). After you decide, your friend can either accept the offer or decline it. If they decline it, you both get nothing. How would you split the chocolate coins?

   Answers may vary. There are two cases you must consider:

   (a) Your friend wants as many chocolate coins as possible.

   (b) Your friend wants you to share the coins fairly.

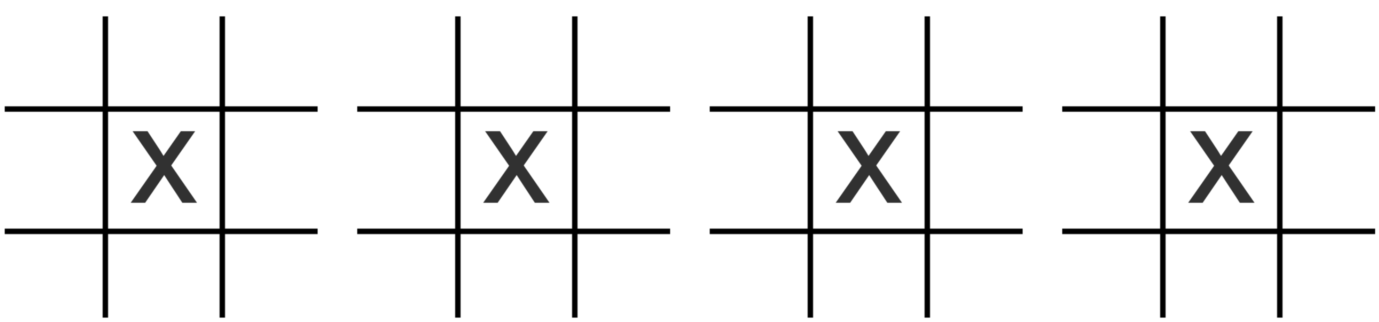
   If the first case is true, then you can give your friend only 1 chocolate coin and keep 9 for yourself, because you know they would rather have 1 chocolate coin than 0.

   If the second case is true, then you should give your friend 5 chocolate coins and keep 5 for yourself, because you know if you do not do this, then you will get 0 chocolate coins.

   This experiment was performed with young kids. If you would like to see a similar one, follow the link here:
   `http://mindyourdecisions.com/blog/2009/11/03/the-ultimatum-game-played-by-children/#.VFKAIcZqp_Q`

7. In the game tic-tac-toe, each player takes a turn to place an X or an O in one of nine spots. The player that forms a straight line of three X's or a straight line of three O's is the winner. The first player usually starts by placing an X in the middle spot on the grid.

   What are all of the spots player 2 can place an O next to give player 1 a winning strategy?



   Any spot but the corner spots.

8. **Twenty-One**

In the game twenty-one, you and a partner take turns subtracting numbers from 21 until one of you reaches zero. Each player can only subtract a number from 1 to 4 during each turn, and the person who reaches zero is the winner.

Play this game with a partner twice, alternating who goes first. Record your numbers in the tables below:

Game 1

You:																		
Partner:																		

Game 2

You:																		
Partner:																		

If you are the first player, there is a way you can win this game every time. What is this winning strategy?

(a) Subtract 1 as your first move.

(b) Choose a number that subtracts a total of 5 when combined with your partner's move each time.

9. * From the game above

(a) What would be the winning strategy if the game started at 24 instead of 21.

Subtract 4 as your first move instead of 1, then repeat subtracting by 5.

(b) Who would have the winning strategy if the game started at 20.

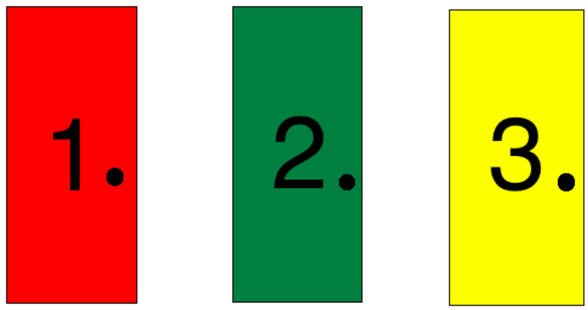
The player with the second move.

(c) If you are only allowed to subtract 1 and 2 each time and the game starts at 21, who has the winning strategy and what is it?

The player with the second move has the winning strategy, and it would be to subtract a total of 3 each time when combined with the first player's move.

10. * Let's Make a Deal!

In the old game show Let's Make a Deal!, contestants are asked to choose between 3 doors. Behind one of the doors is a brand new car, but behind each of the other two doors is a goat. Once the contestant has chosen a door, one of the other two doors is opened, revealing a goat. The contestant is then given a choice to keep the door that he or she has chosen, or switch to the other remaining door. What should the contestant do?



Play this game with a partner. Repeat the game 6 times with you as the contestant (record your results on this sheet), and 6 times with your partner as the contestant (your partner will record his or her results on his or her sheet). Make sure you and your partner choose a different strategy that you will stick with for all 3 trials (ie. you choose to switch doors every time and your partner chooses to keep the same door everytime.)

Trial #	Door Picked	Door With Goat	Switch/Don't Switch	Win/Lose
1				
2				
3				
4				
5				
6				

What is the better strategy?

The better strategy is to switch doors every time, because it gives you a probability of $\frac{2}{3}$ of winning the car. This is because you originally only had a probability of $\frac{1}{3}$ of choosing the car to begin with, and a probability of $\frac{2}{3}$ that the car was behind one of the other two doors. Since we eliminate one of those other two doors, the probability of $\frac{2}{3}$ shifts to the remaining door.

11. * The Pirate Game

Five pirates were sailing one day and stumbled upon a treasure chest filled with 10 gold coins. The captain, Nash, and pirates 2, 3, 4, and 5 had to decide how the gold was to be shared. Being the captain, Nash was the first one to make a decision. The rules and conditions of the game are as follows:

Rules and Conditions:

- Nash is to propose how the pirates will share the gold, and the rest of them vote whether or not they agree. If Nash gets at least 50% of the vote in his favour, the gold will be shared his way (Nash's vote counts).
- If the vote is less than 50%, then Nash is thrown off the ship and pirate 2 becomes the captain and the game is repeated.
- The pirates' first goal is to remain on the ship and their second goal is to maximize the amount of gold coins they get.
- Assume that all 5 pirates are intelligent, rational, greedy, and do not wish to be thrown off the ship.

Nash finds a plan to maximize his gold and stay alive. What is the plan? (Hint: work backwards from the situation if it was just pirates 4 and 5 on the ship. What's the offer pirate 4 can make? What offer must pirate 5 accept before he is left alone with pirate 4?

Answers may vary, but there is the optimal solution:

The way we find the optimal solution is by working backwards to see if we can find any situations where a pirate will accept a low amount of gold. Remember, Nash wants to give away the least amount of gold coins as possible. If we can find these situations, then Nash, being an intelligent pirate, will also be able to find them.

Assume that pirates Nash, 2, and 3 have been thrown off the ship after their offers of how to share the gold were rejected. We are then left with pirates 4 and 5:

4	10
5	0

In this situation, pirate 4 can take all 10 gold coins for himself and still get 50% of the votes, since his vote counts. Therefore, pirate 5, being greedy, must do what is necessary to avoid this situation where its just him and pirate 4 on the ship. So, now, let's assume pirate 3 is back on the ship:

3	9
4	0
5	1

In this situation, pirate 3 can offer pirate 5 only 1 gold coin, because, being intelligent, he knows that if pirate 5 does not accept this offer then pirate 5 will get 0 gold coins when it is pirate 4's turn to make an offer. So pirate 5 will accept this offer, being rational, and pirate 3 will get at least 50% of the vote. However, pirate 4, being greedy, looks at this situation and realizes that he must do what is necessary to avoid it, or else he will end up with zero gold coins. So, now, let's assume pirate 2 is back on the ship:

2	9
3	0
4	1
5	0

In this situation, pirate 2 can offer pirate pirate 4 only 1 gold coin, because, being intelligent, he knows that if pirate 4 does not accept this offer then pirate 4 will get 0 gold coins when it is pirate 3's turn to make an offer. So pirate 4 will accept this offer, being rational, and pirate 2 will get at least 50% of the vote. However, pirates 3 and 5 look at this situation and realize that they must do what is necessary to avoid it, or else they will end up with zero gold coins. So, now, let's assume pirate Nash is back on the ship:

Nash	8
2	0
3	1
4	0
5	1

In this situation, pirate Nash can offer pirates 3 and 5 each one gold coin, because, being intelligent, he knows that if pirates 3 and 5 do not accept this offer then they will both get 0 gold coins when it is pirate 2's turn to make an offer. So pirates 3 and 5 will accept this offer, being rational, and pirate Nash will get at least 50% of the vote. Therefore, Nash should offer pirates 3 and 5 one gold coin each and keep eight for himself.