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Grade 7/8 Math Circles
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Sequences

Sequences

A **sequence** is a list of objects that follow a certain order. Each object in the list is called a term, and we refer to it using subscript notations as a_n, where n is a number that indicates its position in the sequence. We often put curly braces "{" around our list to indicate that it is a sequence.

For example,

$$\{1, 5, 9, 13, 17, \dots\} \tag{1}$$

has terms $a_1 = 1$, $a_2 = 5$, $a_3 = 9$, $\dots$ with the rule that every element is four greater than the previous one starting at 1. This sequence can be modeled by

$$a_n = 4 + a_{n-1} \quad \textit{with} \quad a_1 = 1$$

There are two types of sequences to know, **arithmetic** and **geometric**.

Arithmetic Sequence

An arithmetic sequence is a sequence where the difference between terms is constant. We will call this the common difference. How do you find this difference? Take the difference of any two side by side terms. The sequence from above is an arithmetic sequence as you can take any two side by side terms and their difference is 4.

Which of the following are arithmetic sequences:

- $\{1, 2, 3, 4, 5, \dots\}$
- $\{2, 4, 8, 16, \dots\}$
- $\{2, 4, 5, 8, 14, 19\}$
- $\{3, 5, 7, 9, 11, 13, \dots\}$
- $\{3, 3, 3, 3, 3, 3, 3\}$
- $\{10, 110, 210, 310, \dots\}$
- $\{7, 2, -3, -8, -13, \dots\}$
- $\{-77, -66, -55, -44, \dots\}$

Defining a Sequence

The **recursive definition** of the sequences is is a mathematical expression that depends on the previous term in the sequence. We saw one earlier when we said $a_n = 4 + a_{n-1}$.

Is something missing? Yes, the first term. In a recursive definition, since each term depends on the previous, you need to specify what the first term is. That is why earlier we wrote $a_n = 4 + a_{n-1} \quad with \quad a_1 = 1$. The recursive formula for an arithmetic sequence is

$$a_n = a_{n-1} + d$$

where d is the common difference.

Another way to represent a sequence is in a **closed form** solution. Here the mathematical expression does not depend on the previous term but rather the term number. For example the sequence (1) would be defined as:

$$a_n = 1 + 4(n - 1)$$

The n is the term number in this expression. For example, the 11^{th} term in the sequence is

$$\begin{aligned}
a_n &= 1 + 4(n - 1) \\
a_{11} &= 1 + 4(11 - 1) \\
a_{11} &= 1 + 4(10) \\
a_{11} &= 1 + 40 \\
a_{11} &= 41
\end{aligned}$$

In general the equation for an arithmetic sequence is:

$$a_n = a + d(n - 1)$$

Here d is the common difference and a is the first term of the sequence.

The benefit of this is that we do not need to know the first term in the definition of the sequence and we do not need the previous term to calculate the next. This can greatly reduce the time to calculate a term. In the recursive definition, if you want the 100th term, you need to calculate the 99th. For the 99th you need to calculate the 98th and so on. Alternatively, to find the 100th term in the equation above you simply replace n with 100 and solve.

Example: Find the closed form solution:

- $\{4, 6, 8, 10, \dots\}$
- $\{5, 12, 19, 26, \dots\}$
- $\{51, 42, 33, 24, \dots\}$

Geometric Sequence

A geometric sequence is a sequence where each term is found by multiplying the previous term by a constant number known as the common ratio. How do you find the common ratio? Take any number from the sequence and divide it by the number right before it. For example:

$$\{3, 6, 12, 24, 48, \dots\} \tag{2}$$

The common ratio here is $\frac{12}{6} = 2$. Therefore this is a geometric sequence with the rule, each term is two times the previous starting at 2. Similarly to how we modeled the previous sequence with a recursive definition, we can do the same with this one.

$$a_n = 2a_{n-1} \quad with \quad a_1 = 3$$

The recursive definition of a geometric sequence can be generalized as:

$$a_n = r \times a_{n-1}$$

Here r is the common ratio.

Examples Find the recursive definition

- $\{2, 8, 32, \dots\}$
- $\{7, 49, 343, 2401, \dots\}$
- $\{1, -2, 4, -8, 16, \dots\}$

Examples For the following sequences state if they are a geometric sequence, if so give its recursive definition.

- $\{1, 2, 3, 4, 5, \dots\}$
- $\{2, 4, 8, 16, \dots\}$
- $\{40, 20, 10, 5, \dots\}$
- $\{3, 9, 36, 180, \dots\}$
- $\{3, 3, 3, 3, 3, 3, 3\}$
- $\{100, 1000, 10000, 100000, \dots\}$
- $\{7, 2, -3, -8, -13, \dots\}$
- $\{-5, 15, -45, 135, \dots\}$

You may have noticed we used a multiplication to express an arithmetic sequence in closed form. This is because repeatedly adding a number is the definition of a multiplication. For a geometric sequence we are repeatedly doing multiplication, which is the definition of an exponent. So, the sequence (2) can be expressed as:

$$a_n = 3 \times 2^{n-1}$$

In general the closed form for a geometric equation is:

$$a \times r^{n-1}$$

where a is the first term, r is the common ratio and n is the term number

Examples:

Find the closed form solution for the following sequences.

- $\{2, 8, 32, \dots\}$
- $\{2, 8, 32, \dots\}$
- $\{40, 20, 10, 5, \dots\}$

For the following sequences identify the type of sequence, write the recursive definition as well as the closed form and finally find the 8th term in the sequence:

- $a_n = \{5, 10, 15, 20, 25, \dots\}$
- $a_n = \{8, 24, 72, 216, 648, \dots\}$
- $a_n = \{5, 10, 20, \dots\}$
- $a_n = \{1, 10, 19, 28, \dots\}$

Finding the Term Number

Now that we know how to find the n^{th} number in a sequence we can look at finding what term corresponds to a given number. For example the sequence: $\{4, 9, 14, 19, \dots\}$ can be expressed as $a_n = 4 + 5(n - 1)$. Given this, what term in the sequence will have the value 124? How do we approach this? Similar to before,but now we replace a_n instead of n.

$$\begin{aligned}
a_n &= 4 + 5(n-1) \\
124 &= 4 + 5(n-1) \\
124 &= 4 + 5n - 5 \\
124 &= 5n - 1 \\
124 + 1 &= 5n - 1 + 1 \\
125 &= 5n \\
\frac{125}{5} &= \frac{5n}{5} \\
n &= 25
\end{aligned}$$

We say that the 25th terms will have the value 124.

Find the term in sequence $a_n = 7 + 8(n-1)$ equal to 63.

$$\begin{aligned} a_n &= 7 + 8(n-1) \\ 63 &= 7 + 8(n-1) \\ 63 - 7 &= 7 + 8(n-1) - 7 \\ 56 &= 8(n-1) \\ \frac{56}{8} &= \frac{8(n-1)}{8} \\ 7 &= n - 1 \\ 7 + 1 &= n - 1 + 1 \\ n &= 6 \end{aligned}$$

The 6th term in the sequence is equal to 63.

Examples

Find which term in the sequence is equal to 150.

1. $a_n = 60 + 3n$
2. $a_n = 30 + 6(n-1)$

Find the term in the sequence equal to 84

1. $a_n = \{2, 5, 8, 11, 14, \ldots\}$
2. $a_n = \{21, 28, 35, \ldots\}$

PROBLEMS

1. Identify the type, find the rule, write the recursive definition and the non recursive definition of the following sequences.

   (a) $a_n = \{2, 4, 6, 8, \ldots\}$

   - Arithmetic
   - Add 2 to each number starting at 2
   - $a_n = a_{n-1} + 2, \quad a_1 = 2$
   - $a_n = 2n$

   (b) $a_n = \{4, 23, 32, \ldots\}$

- Arithmetic
- Add 19 to each number starting at 4
- $a_n = a_{n-1} + 19 \quad a_1 = 4$
- $a_n = 19n - 15$

(c) $a_n = \{12, 132, 1452, \dots\}$

- Geometric
- Multiply each number by 11 starting at 12
- $a_n = 11a_{n-1} \quad a_1 = 12$
- $a_n = 12(11)^{n-1}$

(d) $a_n = \{2, 4, 8, 16, \dots\}$

- Geometric
- multiply each number by 2 starting at 2
- $a_n = 2a_{n-1} \quad a_1 = 2$
- $a_n = 2^n$

2. Find the pattern. You can describe it anyway you like.(note some are not not arithmetic or geometric)

(a) $a_n = \{12, 20, 28, \dots\}$

$8n + 4$

(b) $a_n = \{2, 3, 6, 18, 108, \dots\}$

$a_n = a_{n-1}a_{n-2}$

(c) $a_n = \{3, -12, 36, -144, \dots\}$

$3(-4)^{n-1}$

(d) $a_n = \{3, 5, 10, 12, 24, 26, \dots\}$

Alternate between adding 2 and 5, starting at 3

3. Find the 13th term of $\{3, 7, 11, 15, \dots\}$

$a_n = 4n - 1$

$a_n = 4(13) - 1$

$a_n = 51$

4. Find the 8th term of $\{10, 3, -4, \dots\}$

   $a_n = 10 - 7(n-1)$

   $a_n = 10 - 7(8-1)$

   $a_n = -39$

5. Find the 21st term of $\{16, 8, 4, 2, \dots\}$

   $a_n = 16 \times \frac{1}{2}^{n-1}$

   $a_n = 16 \times \frac{1}{2}^{20}$

   $a_n = \frac{1}{2^{16}}$

6. How many terms are in $\{2, 10, 18, \dots, 602\}$

   $a_n = 8n - 6$

   $602 = 8n - 6$

   $608 = 8n$

   $n = 76$

7. How many terms are in $\{13, 20, 27, \dots, 97\}$

   $a_n = 13 + 7(n-1)$

   $97 = 13 + 7(n-1)$

   $n = 13$

8. How many terms are in $\{10, 4, -2, \dots, -56\}$

   $a_n = 10 - 6(n-1)$

   $-56 = 10 - 6(n-1)$

   $n = 12$

9. A theater has 502 seats the last row, 494 in the second last row, 486 in the third last row and 22 in the first row. How many rows are in this theater?
   we have $a_n = \{502, 494, 486, \dots, 22\}$ and we can find $d = -8$

$a_n = 502 - 8(n-1)$

$22 = 502 - 8(n-1)$

$22 - 502 = 502 - 8(n-1) - 502$

$-480 = -8(n-1)$

$60 = n - 1$

$n = 61$

There are 61 rows in this theater

10. A pyramid is built of blocks stacked on each other. If the top of the pyramid is a single block, the level below it is 4 blocks, and the level below that is 9 blocks, and one below that is 16 blocks how many blocks would be on the bottom level is there are 50 levels? (find the closed form solution then calculate the answer)
$\{1, 4, 9, 16, \dots\} = \{1^2, 2^2, 3^2, 4^2, \dots\}$ and this has a closed form of n^2 the the bottom level has $50^2 = 2500$ blocks

11. If you are given a sequence and a number, how could you tell if that number is in the sequence? (without out writing out the whole sequence)
Assume it is in the sequence and try to solve for n. If n is an integer then that is the term number, if n isn't an integer then the number is not in the sequence

12. Given the sequence $\{2, 5, 8, 11, \dots, 602\}$, is 321 in this sequence? If it is what term corresponds to this number?

$a_n = 3n - 1$

$321 = 3n - 1$

$322 = 3n$

322 is not divisible by 3 and therefore 321 is not in the sequence

13. ***You notice you have no pens and you see a great deal on bulk pens. so you sign up for a subscription that sends you 17 pens every month. Once you reach 119, you realize you have way to many pens and start giving away 24 a month to your friends (your subscription is still active). At the end of your subscription you are left with 42 pens. How long was your subscription?

First we find out how many weeks until you hit 119 pens. We can write this as $a_n = 17 + 17 \times (n-1)$(note $a = 17$ because we have 17 after week 1. We first solve,$17 + 17 \times (n-1) = 119$ and get $n = 7$.

Now at this point you are receiving 17 pens and giving 24 away every month. This means that each month your pen total decreases by 7 ($17 - 24 = -7$). Therefore our new sequence is $112 - 7(n-1)$ and we want to solve $112 - 7(n-1) = 42$.
Doing so you get $n = 11$

Now add the two values of n you found $7+11 = 18$. You subscription is 18 weeks long.

14. ***find the last digit of the 43rd term in $\{7^1, 7^2, 7^3, 7^4\}$ **without using a calculator** (Hint: Calculate the first few and see if there is a pattern.)

    Looking at the first few terms we have $\{7, 49, 343, 2401, 16807, 117649, \dots\}$
    we notice that the last digit follows the pattern $\{7, 9, 3, 1, 7, 9, 3, \dots\}$ we notice that the 4th,8th,12th term end in 1, these are multiples of 4. 44 is a multiple of 4 so it will also end in 1, however we are looking for the digit of 43 so we take the number that comes before 1 and that is 3.

15. ***Find the next 2 terms after $\{1, 7, 21, 46, 85, 141, \dots\}$ (Hint: Think about the differences between the terms, also this is VERY difficult)

    First look at the differences between terms. We get
    $\{6, 14, 25, 39, 56, \dots\}$
    Now take the deference between terms again and you get
    $\{8, 11, 14, 17, \dots\}$ This sequence has an easy pattern or $a_{n-1} + 3$. the next term in this sequence is 20. Since these are the differences of the previous sequence we know the next number in $\{6, 14, 25, 39, 56, \dots\}$ is $56+20 = 76$.
    Now 76 is part of the sequence that is described as the difference between terms of the original sequence. This make the next number in our original sequence 141+76=217.

    Following the same steps with the new sequence
    $\{1, 7, 21, 46, 85, 141, 217, \dots\}$
    will give you the second number we are looking for, 316.