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Grade 7/8 Math Circles

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Series

Sequence Recap

Before starting series lets recap last weeks lesson and look at one special sequence.

A **sequence** is a list of objects that follow a certain order. Each object in the list is called a term, and we refer to it using subscript notations as a_n, where n is a number that indicates its position in the sequence. For example:

$$\{5, 8, 11, 14, 17, 20, \dots\}$$

In this sequence we have $a_1 = 5, \quad a_2 = 8, \quad a_3 = 11$ and so on. We can define this sequence recursively as:

$$a_n = a_{n-1} + 3 \quad a_1 = 5$$

Or in closed form as:

$$a_n = 5 + 3(n - 1)$$

We covered two types of sequences, arithmetic and geometric. Here they are summarized:

- Arithmetic
  - There is a common difference between each term.
  - Each term is defined as the common difference(d) plus the previous term.
  - The recursive definition is: $a_n = a_{n-1} + d$
  - The closed form solution is: $a_n = a + d(n - 1)$
- Geometric
  - There is a common ratio between each term

- Each term is defined as the previous number multiplied by the common ratio(r)
- The recursive definition is: $a_n = r \times a_{n-1}$
- The closed form solution is: $a_n = a \times r^{n-1}$

New Topics:

Fibonacci sequence

The Fibonacci sequence is an infinite sequence(goes on for ever) that is very famous in the mathematical world. Here is the beginning of the sequence, can you find the pattern?

$$\{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$$

The Fibonacci sequence is defined as:

$$F_n = F_{n-1} + F_{n-2}$$

Why is is this sequence so famous??? Because of its mysterious relation to nature. For example, number of petals on a flower is typically a Fibonacci number. Some shells grow in a Fibonacci spiral. It is an interesting sequence mathematically speaking, however it does not have many practical applications.

Series

A series is the sum of the terms of a sequence. We denote a series as S_n. Another way of writing a series is with the summation symbol sigma($\sum$).

$$\sum_{i=1}^{N} a_i = a_1 + a_2 + \cdots + a_{n-1} + a_n$$

In this notation the $\sum$ indicates adding all the a_i from $i = 1$ to $i = N$. We can also do this for closed form solutions. For example:

$$\sum_{n=1}^{6} 1 + 6(n-1) = [1 + 6(1-1)] + [1 + 6(2-1)] + \cdots + [1 + 6(5-1)] + [1 + 6(6-1)]$$

$$\sum_{n=1}^{6} 1 + 6(n-1) = 1 + 7 + 13 + 19 + 25 + 31$$
$$= 8 + 32 + 56$$
$$= 96$$

Examples

Find the sum of the following series:

1. $S_n = 1 + 2 + 3 + 4 + 5$
2. $S_n = 3 + 8 + 13 + 18$
3. $S_n = 2 + 4 + 8 + 16 + 32 + 64 + 128$
4. $a_i = \{5, 7, 13, 24, 25, 95\} \quad find \quad \sum_{i=1}^{4} a_i$
5. $\sum_{n=0}^{5} 9n$

A Faster Way?

Find the sum of of the first 100 numbers. That is:

$$S = \sum_{n=1}^{100} n = ?$$

$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + \dots$

I give up...

trick:

$$\begin{array}{rrrrrrrr} & 1\ + & 2\ + & 3\ + & \dots & +\ \ 98\ + & 99\ + & 100 \\ + & 100\ + & 99\ + & 98\ + & \dots & +\ \ \ 3\ + & 2\ + & 1 \\ \hline & 101\ + & 101\ + & 101\ + & \dots & +\ 101\ + & 101\ + & 101 \end{array}$$

$2S = 101(100)$

$S \ = \frac{101(100)}{2} = 5050$

This can be generalized to numbers from 1 to N by the formula

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2}$$

Find the sum of the first 900 positive numbers:

Using the formula we get:

$$\begin{aligned}\frac{N(N+1)}{2} &= \frac{900(900+1)}{2} \\ &= \frac{900(901)}{2} \\ &= \frac{810900}{2} \\ &= 405450\end{aligned} \tag{1}$$

Imagine how long that would take you to calculate if you did not know the trick(formula). Lets do an other example.

Find the sum of the first 43 positive numbers;

$$\begin{aligned}\frac{N(N+1)}{2} &= \frac{43(43+1)}{2} \\ &= \frac{43(44)}{2} \\ &= \frac{1892}{2} \\ &= 946\end{aligned}$$

Other formula used for series:

What technique did we use for the sum of the first n numbers? We inverted the series and added it up with itself. What if the series doesn't start at 1? Does this still work? Yes and there is only a slight modification.

We will look at the generalized formula in two different ways:

The first formula uses the arithmetic sequence formula $a + d(n-1)$, lets look at this using the same adding trick.

$$\begin{array}{cccccccc} & a\,+ & a+d\,+ & \dots & + & a+d(n-2)\,+ & a+d(n-1) \\ + & a+d(n-1)\,+ & a+d(n-2)\,+ & \dots & + & a+d\,+ & a \\ \hline & 2a+d(n-1)\,+ & 2a+d(n-1)\,+ & \dots & + & 2a+d(n-1)\,+ & 2a+d(n-1) \end{array}$$

Adding these terms together we get

$$S_n = \frac{n[2a + d(n-1)]}{2}$$

The second way of looking at this formula is looking at what we did when we added 101 with itself 100 times earlier. The 101 came from adding $1 + 100$ or we can see it as adding the first number with the last. Lets look at the addition trick we did before but this time instead of 1 to 100 we will do a_1 to a_n

$$\begin{array}{ccccccccccccc} & a_1 + & a_1 + d + & a_1 + d + d + & \ldots & + & a_n - d - d + & a_n - d + & a_n \\ + & a_n + & a_n - d + & a_n - d - d + & \ldots & + & a_1 + d + d + & a_1 + d + & a_1 \\ \hline & (a_1 + a_n) + & (a_1 + a_n) + & (a_1 + a_n) + & \ldots & + & (a_1 + a_n) + & (a_1 + a_n) + & (a_1 + a_n) \end{array}$$

Using the same logic as before we can generalize the sum of first n numbers to:

$$S_n = \frac{n(a_1 + a_n)}{2}$$

Here n is the number of terms in the series, a_1 is the first number and a_n is the last term in the series. These two formulas give the exact same result they are simply written in two different ways.

Examples:

Given the sequence $a_n = 9 + 2(n-1)$ find the sum of the first 15 terms.

lets solve this using the formula $\frac{n(2a + d(n-1))}{2}$

$$S_n = \frac{n(2a + d(n-1))}{2}$$
$$S_n = \frac{15(2a + d(n-1))}{2}$$
$$S_n = \frac{15(2(9) + d(n-1))}{2}$$
$$S_n = \frac{15(2(9) + 2(n-1))}{2}$$
$$S_n = \frac{15(2(9) + 2(15-1))}{2}$$
$$S_n = 345$$

Given the sequence $a_n = 6 + 4(n-1)$ find the sum of terms a_2 to a_{21}

First we note that in our series we have $21 - 2 + 1 = 20$ term, so $n = 20$

Second we find the first term of our series which is a_2

$$a_2 = 6 + 4(2 - 1)$$
$$a_2 = 6 + 4(1)$$
$$a_2 = 10$$

Now we find the last term of our series a_{21}

$$a_{21} = 6 + 4(21 - 1)$$
$$a_{21} = 6 + 4(20)$$
$$a_{21} = 6 + 80$$
$$a_{21} = 86$$

Once we have these numbers we can use our formula.

$$\begin{aligned}\frac{n(a_2 + a_{21})}{2} &= \frac{20(86 + 10)}{2} \\ &= \frac{20(96)}{2} \\ &= \frac{1920}{2} \\ &= 960\end{aligned}$$

Find the sum of the following series:

1. $S_n = \{4 + 8 + 12 + 16 + 20 + 24\}$
2. $S_n = \{4 + 10 + 16 + \cdots + 124\}$

PROBLEMS

1. Find the 7th term term of $a_n = 7 + 12(n - 1)$

   $a_n = 7 + 12(n - 1)$

   $a_7 = 7 + 12(7 - 1)$

   $a_7 = 7 + 72$

$a_7 = 79$

2. How many terms are in $\{940, 900, 860, \ldots, 540\}$

   $a_n = 940 - 40(n-1)$

   $540 = 940 - 40(n-1)$

   $-400 = -40(n-1)$

   $10 = (n-1)$

   $n = 11$

3. What is the 10th Fibonacci number?

   55

4. Evaluate the following:

   (a) $S_n = \{3 + 6 + 7 + 9\}$

   25

   (b) $\sum_{n=1}^{5} 2n$

   $2 + 4 + 6 + 8 + 10 = 30$

   (c) $\sum_{n=1}^{8} 7$

   $7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 = 7 \times 8 = 56$

   (d) $\sum_{n=5}^{10} 2 + 3(n-1)$

   $14 + 17 + 20 + 23 + 26 + 29 = 129$

5. Find the sum of all integers from 47 to 93.

   $S_n = \frac{n(a_1+a_n)}{2}$

   $S_n = \frac{47(47+93)}{2}$

   $S_n = \frac{47(140)}{2}$

   $S_n = \frac{6580}{2}$

   $S_n = 3290$

6. Evaluate the following:

(a) $\sum_{n=1}^{56} 4n + 1$

$S_n = \frac{n(a_1+a_n)}{2}$

$S_n = \frac{56(5+225)}{2}$

$S_n = \frac{56(230)}{2}$

$S_n = 2990$

(b) $S_n = \{4 + 12 + 20 + 28 + \cdots + 92\}$

$S_n = \frac{n(a_1+a_n)}{2}$

$S_n = \frac{12(4+92)}{2}$

$S_n = \frac{12(96)}{2}$

$S_n = 576$

(c) $a_n = \{6, 9, 12, 15, \ldots, 33\}$ Find the sum of all numbers in a_n

$S_n = \frac{n(a_1+a_n)}{2}$

$S_n = \frac{10(6+33)}{2}$

$S_n = \frac{10(39)}{2}$

$S_n = 195$

(d) $\sum_{n=1}^{20} 5 - 2(n-1)$

$S_n = \frac{n[2a+d(n-1)]}{2}$

$S_n = \frac{20[2(5)-2(20-1)]}{2}$

$S_n = \frac{20[10-38]}{2}$

$S_n = -280$

7. Given the sequence $a_n = 1 + 4(n-1)$ what is the sum of every second number in the sequence up until the 16th number. That is $a_2 + a_4 + a_6 + \ldots a_{14} + a_{16}$.

In the original sequence we have $d = 4$ however we are interested in every second number so we can express that as a new sequence with $a_n = 5 + 8(n-1)$ From there we can use the formula:

$$S_n = \frac{n[2a + d(n-1)]}{2}$$

$$S_n = \frac{8[2 \times 5 + 8(8-1)]}{2}$$

$$S_n = \frac{8[10 + 8(7)]}{2}$$

$S_n = \frac{8[66]}{2}$
$S_n = \frac{528}{2}$
$S_n = 264$

8. In the first row of a theater there are 15 seats, and in the 16th and final row there are 60 seats. If the seat follow an arithmetic sequence:

   (a) What is the common difference?

   $a_n = a + d(n-1)$

   $60 = 15 + d(15)$

   $d = 3$

   (b) How many seats are there in this theater?

   $S_n = \frac{n(a_1 + a_n)}{2}$
   $S_n = \frac{16(15+60)}{2}$
   $S_n = 600$

   (c) * Some strange accident happens and every 3rd row is now unavailable. That is rows 1,2,4,5,7,8,... are available but 3,6,9,... are not. How many people can the theater seat now?

   The seats that are unavailable follow the sequence $a_n = 21 + 9(n-1)$ where $n = 5$
   $S_n = \frac{n(2a + d(n-1))}{2}$
   $S_n = \frac{5(42 + 9(4))}{2}$
   $S_n = 195$

   So there are $600 - 195 = 405$ *seats available*

9. Explain the pattern in the sequence $\{1, 1, 4, 9, 25, 64, 169\}$

   These are the Fibonacci numbers squared

10. **Give a mathematical recursive definition of for the sequence above.

   $F_n = (\sqrt{F_{n-1}} + \sqrt{F_{n-1}})^2$

11. **You are given an arithmetic series whose first number is 4 its last number is 59 and its sum is 189. What is the common difference between the numbers?
We first find the number of terms the sequence by using $\frac{n(a_1 + a_n)}{2}$
$189 = \frac{n(5+59)}{2}$
$n = 6$
From here we can use a second formula $a_n = 5 + d(n-1)$
$59 = 4 + d(6-1)$
$d = 11$ The common difference in this series is 11.

The following questions deal with geometric series. Only try these if you have finished all the questions above. Note if you missed it in the presentation and want to try these problems the sum of a geometric series is: $\sum_{n=1}^{N} ar^{n-1} = a\frac{1-r^N}{1-r}$

EXTRA NOTE: These questions are extremely difficult and were made for the students who finish all the above question very quickly. Many of these questions are way above a grade 7/8 level. Don't be discouraged if you cannot solve these.

1. Calculate $\sum_{n=1}^{8} 7(2)^{n-1}$

   Using the formula above we need to find a, r and N
   In this example we have a=7, r=2, and n=8. Replace this in our formula and get:
   $7\frac{1-2^8}{1-2} = 1785$

2. Calculate $\sum_{n=1}^{12} 2(3)^{n-1}$

   $2\frac{1-3^12}{1-3} = 531\ 440$

3. Calculate $\sum_{n=1}^{5} 10(4)^{n-1}$

   $10\frac{1-4^5}{1-4} = 3410$

4. *You are offered a job for the month of November working Monday to Friday each week. Your boss offers you to salary options.
   Option 1: you are payed \$1000 everyday of November.
   Option 2: You start at a penny a day(\$0.01) and the amount you are payed doubles

each day. Which option do you take? Calculate how much each make during the 30 days.

You should take the second option.

We have a geometric series with a = 0.01, r = 2 and n = 30

$S_n = a\frac{1-r^n}{1-r}$

$S_n = 0.01\frac{1-2^{30}}{1-2}$

$S_n = 0.01\frac{-1073741823}{-1}$

$S_n = 10,737,418.23$

The second method pays just under 11 million dollars during the month.

5. *These two questions are geometric series that start at a term other then 1, because of this they use a slightly different formula $\sum_{n=j}^{N} ar^{n-1} = a\frac{r^{j-1}-r^N}{1-r}$

   (a) Calculate $\sum_{n=3}^{8} 7(2)^{n-1}$

   $$7\frac{2^2-2^8}{2-1} = 1764$$

   (b) Calculate $\sum_{n=5}^{12} 2(3)^{n-1}$

   $$2\frac{3^4-3^{12}}{3-1} = 531\ 360$$

6. ***A fraction less then 1 to a very large power $(\frac{1}{2}^{10000})$ becomes so small in math we say it converges to 0. As you can see in the sequence here $\{0.5, 0.25, 0.125, \ldots, 0.0009, \ldots\}$. 0.0009 is only the 10th term, imagine how small it will be for the 10,000th term. This is why for calculation we consider $\frac{1}{2}^{some\ really\ big\ number}$ as 0 when we do calculations.

   Knowing this does the infinite series $\sum_{n=1}^{\infty} \frac{1}{2}^n$ have a finite sum. That does it have a sum less then infinity? If so What is that sum?

   Yes the infinite series here has a finite sum.

   $S_n = a\frac{1-r^n}{1-r}$

   Since this is an infinite series, n = ∞ and is therefore large enough to say $\frac{1}{2}^n = \frac{1}{2}^\infty$ converges to 0.

   $S_n = a\frac{1-0}{1-r}$

   $S_n = a\frac{1}{1-r}$

   $S_n = \frac{a}{1-r}$

$S_n = \frac{1/2}{1-(1/2)}$
$S_n = \frac{1/2}{1/2}$
$S_n = 1$

7. ** In an infinite geometric series, what values of r will give you a finite sum?
For any values of r less than 1 but greater then -1. That is $-1 < r < 1$