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Grade 7/8 Math Circles

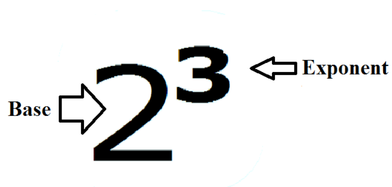
October 7/8, 2014

Exponents and Roots - SOLUTIONS

This file has all the missing parts from the lesson as well as the answers to the problems at the end.

Exponents

An **exponentiation** is a repeated multiplication. Similar to how a multiplication is a repeated addition. Remember, 5×3 is simply $5 + 5 + 5$. Similarly an exponentiation, 5^3 is simply $5 \times 5 \times 5$.



As shown in the picture above, we call the "lower" number the **base**, the "upper" number the **exponent** and when refering to the base and exponent as a whole we will say the **power**. When we see this notation we say "Two to the exponent three". Note: the second and third exponents are often referred to as squared and cubed, respectively. So we might say "two cubed" instead of "Two to the exponent three".

Examples:

Evaluate the following.

1. $2^3 = 2 \times 2 \times 2 = 8$
2. 5^2
3. 3^4

4. 10^3

5. 5^4

Write the following numbers as exponents with the given bases

1. 32, base 2 $= 2^5$

2. 81, base 9

3. 81, base 3

4. 100, base 10

5. 1, base 4

Special Cases

Base 10: what is 10^7? How about 10^n?

Base 10 powers are 1 followed by n zeros, where n is the exponent.

The first power: What is 5^1? How about 123456789^1?

Any number raised to the exponent of 1 is equal to the base.

The power of zero: What is 8^0? How about 9384712^0?

Any non-zero number raised to the exponent of 0 is equal to 1

Now that we have covered the basics of exponents we can look at operations on exponents.

Multiplication

Since a power is simply a repeated multiplication it would only be natural to have rules for multiplying and dividing powers.
How could we simplify :

2×2

This one is easy it is 2^2

Do you agree that the above could have been written as $2^1 \times 2^1$? What can we say about the exponents when looking at the following equality?

$2^1 \times 2^1 = 2^2$

It looks like we are adding the exponents. Consider the multiplication $2^5 \times 2^3$.

$2^5 \times 2^3$	This can be written as
$2 \times 2 \times 2 \times 2 \times 2 \times 2^3$	Again as
$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$	This is 8 2's multiplied together, it is also 2^8. So we get
$2^5 \times 2^3 = 2^8$	

Rule:The multiplication of 2 powers with the same base is simplified to that same base whose exponent is the sum of the 2 exponents:

$$a^m \times a^n = a^{(m+n)}$$

NOTE: THE BASES HAVE TO BE THE SAME

Examples: Simplify to a single exponent if possible

1. $5^3 \times 5^9$
2. $2 \times 2^2 \times 2^3 \times 2^4 \times 2^5$
3. $7^3 \times 7^{13}$
4. $4^7 \times 9^2$
5. $2^4 \times 4^2 \times 2^4$

division:

Consider $\frac{2^5}{2^3}$

$\frac{2^5}{2^3}$	This can be written as
$\frac{2\times2\times2\times2\times2}{2^3}$	Again as
$\frac{2\times2\times2\times2\times2}{2\times2\times2}$	Now with a little work on the fraction we get

$\frac{\cancel{2}\times\cancel{2}\times\cancel{2}\times2\times2}{\cancel{2}\times\cancel{2}\times\cancel{2}}$ Here we cancel out some 2's and we are left with two 2's

$2 \times 2 = 2^2$

$\frac{2^5}{2^3} = 2^2$

Rule:The division of two powers with the same base is simplified to that same base whose exponent is the difference of the 2 exponents:

$$\frac{a^m}{a^n} = a^{(m-n)}$$

Examples:
Simplify the following (if the bases are numbers, give their value)

1. $\frac{4^3}{4^2} = 4$
2. $\frac{7^7}{7^7}$
3. $\frac{3^{300}}{3^{298}}$
4. $\frac{h^{45}}{h^{44}}$

Negative Exponents:

Simplify $\frac{2^5}{2^7}$

According to our previous rules, this gives 2^{-2}. What does this mean?
Looking at this as we did before we see that

$\frac{2^5}{2^7}$ This can be written as

$\frac{2\times2\times2\times2\times2}{2\times2\times2\times2\times2\times2\times2}$ Simplify to get

$\frac{1}{2\times2}$

$\frac{1}{2^2}$

$\frac{1}{2^2} = 2^{-2}$

So a negative exponent in the numerator becomes a positive if it is sent to the denominator. Similarly a negative exponent in the denominator becomes a positive exponent in the numerator. That is, $\frac{1}{2^{-2}} = 2^2$

Examples: Simplify the following. Write the answers with positive exponents.

1. $\frac{8^2}{8^4}$
2. $13^3 13^{-3}$
3. $\frac{8^4}{8^{-4}}$
4. $\frac{3^{-2}}{3^{-3}}$

Power of a Power

What? $(3^4)^5$ is this even legal? Yes, and its not much more than we already covered.

Look at $(3^4)^5$ If we consider the inner exponentiation to simply be a number we can write.

$$3^4 \times 3^4 \times 3^4 \times 3^4 \times 3^4$$

From before we know this to be equal to 3^{20} since $4 + 4 + 4 + 4 + 4 = 20$. We can also see this as $4 \times 5 = 20$. A power raised to a power is simplified by multiplying the exponents.

$$(a^n)^m = a^{n \times m}$$

Extended to more than one power, each exponent gets multiplied.

$$(a^n b^k)^m = a^{n \times m} b^{k \times m}$$

Simplify:

1. $(5^3)^4$

2. $(4^3 4^7)^{20}$

3. $(2 \times 3 \times 4)^3$

4. $(4^3)^{-2}$

5. $(4^{-3})^{-2}$

6. $((6^2)^2)^2$

7. ***$\dfrac{4^2 3^7}{(2^2 3^3)^3}$

Roots

You may have seen the symbol $\sqrt{}$ before, this indicates the **square root**. The square root, in a way, is the opposite of squaring a number. For example, $\sqrt{4}$ asks what number multiplied by itself is 4. Well it's 2. So $\sqrt{4} = 2$. However, not all roots are easily simplified like $\sqrt{5}$ or $\sqrt{12}$ as there is not a whole number that when multiplied by itself equals 5 or 12. Sometimes these roots can be simplified further, like $\sqrt{12} = 2\sqrt{3}$ but we will get to that later.

First lets relate roots to exponents by saying $\sqrt{x} = x^{1/2}$. If you don't see where this comes from, look at the example from the first paragraph on roots. We stated:

$\sqrt{4} = 2$ Let $\sqrt{4} = 4^n$, so we get
$4^n = 2$ We know from before that $4 = 2^2$. So we can now write
$(2^2)^n = 2$ Using what we learned from exponents
$2^{2n} = 2$ Remember 2 can also be seen as 2^1
$2^{2n} = 2^1$ Now we want to solve $2n = 1$. Doing so we find $n = \frac{1}{2}$ and this means
$\sqrt{4} = 4^{1/2}$

Further we say the nth root, written as $\sqrt[n]{}$ is equal to raising the base to the exponent 1/n.

$$\sqrt[n]{x} = x^{1/n}$$

Remember to solve a square root we looked for a number times itself that is equal to the number inside the root. An n^{th} root is the same, but now we are looking for a number that is multiplied by itself n times. You may have seen that $3^4 = 81$, this also means that $\sqrt[4]{81} = 3$.

Example: Evaluate

1. $\sqrt[3]{8} = 2$
2. $\sqrt{36}$
3. $\sqrt[4]{16}$
4. $\sqrt[5]{1}$

Some of these may seem easy, because you might remember what number raised to a power gave that number from the previous section. But how would you know whether the statement $\sqrt[3]{1728} = 12$ is true or not?

Simplifying Roots

We will now look at how to simplify roots, but first me must quickly review prime factorization.

What is a prime number? a number that is greater then 1 and only divisible by 1 and itself (2,3,5,7,11,...) Any number can be written as a product of prime numbers.

$36 = 2 \times 2 \times 3 \times 3$ is the prime factorization of 36
$252 = 2 \times 2 \times 7 \times 9$ is the prime factorization of 252

Simplifying Roots

We will look at two methods of simplifying roots. The first being the perfect square method and the second the prime factorization method.

Perfect Square Method

This method involves looking for a perfect square as a factor of your root. For example : $\sqrt{81} = 9$ we know this because 81 is a perfect square. But what did we really do there?
$\sqrt{81} = 9$
$= \sqrt{9 \times 9}$
$= \sqrt{9} \times \sqrt{9}$
$= 3 \times 3$
$= 9$

Here we used the property: if $a \geq 0$ and $b \geq 0$ then $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

To perform the perfect square method these steps:

1. Try to find a factor that is a perfect square
2. Factor your root using that number
3. Use the property from above ($\sqrt{ab} = \sqrt{a} \times \sqrt{b}$)
4. Simplify the perfect square

Using this technique, lets simplify $\sqrt{28}$

$= \sqrt{4 \times 7}$
$= \sqrt{4} \times \sqrt{7}$ Here we know 4 is a perfect square, so we get
$= 2\sqrt{7}$

How about $\sqrt{75}$?
$= \sqrt{25 \times 3}$
$= \sqrt{25} \times \sqrt{3}$
$= 5\sqrt{3}$

Prime Factorization Method

Even though the perfect square method will always work and is fast, sometimes it might not be obvious to tell if a perfect square is a factor. If this is the case we can use the prime factorization method.

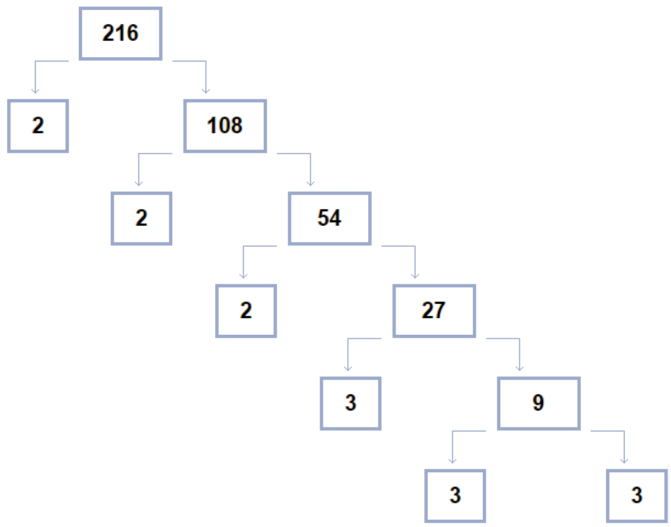
The steps of this new method are:

1. Find its prime factorization
2. Find all the prime that show up n times (n is from your n^{th} root)
3. Remove the primes that are a multiple of n

For example find the cube root and square root of 216:

> 216 can be divided by 36, however this might not be obvious to everyone. 216 does however has a prime factorization of $2 \times 2 \times 2 \times 3 \times 3 \times 3$

That is three 3's and three 2's. If we want the cubed root ($n = 3$) then we can remove one prime to the outside for every three on the inside.



For a cube root we can remove one 2 and one 3 leaving us with $\sqrt[3]{216} = 2 \times 3 = 6$
For a square root, we can also remove a 2 and a 3, but doing so leaves us with a 2 and a 3 still in the root leaving us with $\sqrt{216} = 2 \times 3\sqrt{2 \times 3} = 6\sqrt{6}$

Earlier, I stated that $\sqrt{12} = 2\sqrt{3}$.
12 has prime factorization of $2 \times 2 \times 3$
Since we are looking for a square root, we need two of the same prime in order to take one out. Here we have two 2's and only one 3. Therefore, $\sqrt{12} = 2\sqrt{3}$. We also could have noticed that 12 is 4×3 and 4 is a perfect square.

Examples: Simplify the following roots if possible using either method.

1. $\sqrt{72} = 6\sqrt{2}$
2. $\sqrt[3]{125}$
3. $\sqrt[4]{48}$
4. $\sqrt{111}$

PROBLEMS

1. Write the following as exponents
   (a) $4 \times 4 \times 4$ 4^3
   (b) 7 to the 5 7^5
   (c) $3 \times 3 \times 3 \times 3 \times 3$ 3^5
   (d) 10 10^1
   (e) $8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8$ 8^8
   (f) $8 \times 8 \times 8 \times 8 \times 3 \times 3 \times 3$ $8^4 3^3$
   (g) $9 \times 4 \times 4 \times 9 \times 4$ $9^2 4^3$
2. Evaluate the following:
   (a) 10^6 1000000
   (b) 3^5 243
   (c) 21^0 1
   (d) 71^1 71
   (e) 0^1 0
   (f) $1^0 + 2^0 + 3^0 + 4^0 + 5^0$ 5
   (g) $4^2 + 9^2 - 3^2$ 88
3. Simplify if possible:
   (a) $2^2 \times 2^2$ 2^4
   (b) $3^2 \times 2^3$ $3^2 \times 2^3$
   (c) $5^7 \times 5^7$ $5^1 4$
   (d) $6^4 6^0 6^0$ 6^4
   (e) $4^3 \times 6^5 \times 4^2$ $4^5 6^5$
   (f) $3^3 3^{-3}$ 3^0
   (g) $7^4 7^7 7^{-9}$ 7^2
4. Simplify if possible:

(a) $\frac{3^5}{3^4}$ 3^1

(b) $\frac{7^2}{7^2}$ 7^0

(c) $\frac{8^1}{8^7}$ 8^{-6}

(d) $\frac{3^2}{3^{-2}}$ 3^4

(e) $\frac{1^{738293}}{1^{45802}}$ 1

(f) $\frac{6^{-5}}{6^{-8}}$ 6^2

5. Simplify if possible:

(a) $(4^2)^4$ 4^8

(b) $(3^{12})^0$ $3^0 = 1$

(c) $((4^2)^4)^2$ 4^16

(d) $(7^3 7^2)^3$ 7^15

(e) $(11^2 6^4)^6$ $11^{12}6^{24}$

(f) $\frac{6^3}{(6^{-1})^5}$ 6^8

6. Simplify the following roots:

(a) $\sqrt{5}$ $\sqrt{5}$

(b) $\sqrt{81}$ 9

(c) $\sqrt{70}$ $\sqrt{70}$

(d) $\sqrt{243}$ $9\sqrt{3}$

(e) $\sqrt[5]{243}$ 3

(f) $\sqrt[3]{1125}$ $9\sqrt{9}$

7. If the population of rabbits doubles every year, how many rabbits will there be in 3 years if there are currently 7?

7 after one year 7×2, after 2 years $7 \times 2 \times 2$, after 3 years $7 \times 2 \times 2 \times 2 = 7 \times 2^3$

8. You go back in time and tell you parents to buy into Apple. Your parents wisely listen you and invest \$1000. Since then, The value of apple has tripled 4 times. how much would your parents \$1000 investment be worth now?
   1000 tripled 4 times means $3 \times 3 \times 3 \times 3 = 3^4$ so your parents 1000 dollar investment would be worth $1000 \times 3^4 = 81000$

9. ** $\sqrt{5} \times \sqrt{5} \times 3^{2/3} \times \sqrt[3]{3} =$ 15

10. ** $\dfrac{25^{3/4}}{25^{1/2}} =$ 5

11. ** $\dfrac{3\sqrt{45}}{\sqrt{20}} =$ $\dfrac{9}{2}$

12. ** $\dfrac{\sqrt{14}\sqrt{21}}{\sqrt{54}\sqrt{49}} =$ $\dfrac{1}{3}$

13. ** $\dfrac{x^{2/3} \times y^{3/4} \times z^{4/5}}{z^{2/3} \times y^{4/3} \times x^{4/5}} = \dfrac{z^{2/15}}{x^{2/15} \times y^{5/12}}$

14. ** Express $\dfrac{8^3 \times 32^7}{2^7}$ as a power of base 2.
    2^{37}

15. ** If you have $0 < 10^n < 1\,000\,000\,000$, What is the max value of 3^{-n} ?
    The max value is when $n = 0$, and therefore $3^{-n} = 1$