Question 1

ToyCo, sells two different wooden products: trains and soldiers. They are seeking your help to optimize their weekly production plan. The production of each type of toy requires two types of specialized work (whose availability is in terms of man-hours): basic carpentry, finishing. The following table lists the number of man-hours required for each product, the total available resources and the profit obtained by selling each toy.

	Train	Soldier	Availability
Basic carpentry	2	1	8
Finishing	1	3	9
Profit ($)	3	3	

(a) What decisions must the company make?

Solution

The company must decide how many trains and how many soldiers to produce.

(b) If the company produces 3 trains, what is the total profit?

Solution

The total profit is $3 \times 3 = 9$

(c) If the company produces 2 soldiers, what is the total profit?

Solution

The total profit is $2 \times 3 = 6$

(d) What is the total profit if they produce 3 trains and 2 soldiers?

Solution

The total profit is $3 \times 3 + 2 \times 3 = 15$

(e) If we let

x_1= the number of trains produced

x_2= the number of soldiers produced

Then write an expression to represent the total profit obtained by producing trains and soldiers.

Solution

The total profit is $3x_1 + 3x_2$

x_1 and x_2 are called the *decision variables.* They represent the decisions that one wants to take in the problem.

The expression you wrote for this part is is called an *objective function.* It is a function of the decision variables that you want to maximize or minimize.

(f) Now x_1 and x_2 cannot assume any value we want. There are *Constraints* that must be imposed in their values in order to satisfy the statement of the problem.

Write down what are the constraints involved in this problem.

Solution

The constraints are:

$$2x_1 + x_2 \leq 8 \tag{1}$$

$$x_1 + 3x_2 \leq 9 \tag{2}$$

$$x_1 \geq 0 \tag{3}$$

$$x_2 \geq 0 \tag{4}$$

(g) Find the optimal solution.

Solution

We will do so graphically.

First, note that we can rewrite (1) as

$$x_2 \leq -2x_1 + 8$$

To graph this, we note that we know how to graph

$$x_2 = -2x_1 + 8$$

We draw this line, which corresponds to the red line in Figure 1

Now what we want is the values of x_2 that are below this line.

Note also that we can rewrite (2) as

$$x_2 \leq 3 - \frac{1}{3}x_1$$

To graph this, we note that we know how to graph

$$x_2 = 3 - \frac{1}{3}x_1$$

We draw this line, which corresponds to the green line in Figure 1

Now what we want is the values of x_2 that are below this line.

Finally, note that $x_1 \geq 0$ represents the region to the right of the vertical axis and $x_2 \geq 0$ represents the region above the horizontal axis. Therefore, the feasible region in Figure 1 is the white region, that is, the set of points that is below the green and red line, to the right of the the vertical axis and above the horizontal axis.

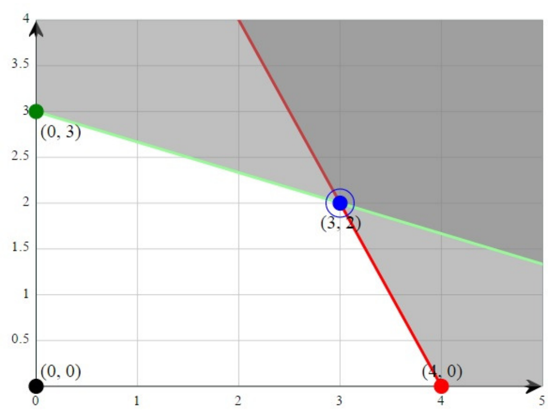


Figure 1: Feasible region

Now, to find the optimal solution, let us look at the objective function $3x_1 + 3x_2$.

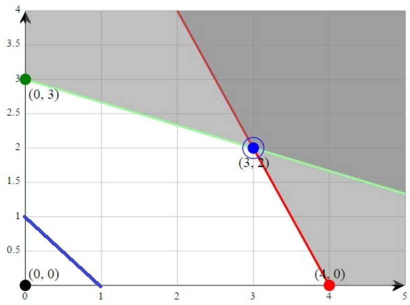
Let us pick an arbitrary value for that objective function, for instance, the value 3. If we look at the plot of the points that satisfy $3x_1 + 3x_2 = 3$ it corresponds to the blue line in figure 2a. Figure 2b shows the blue line that corresponds to points that satisfy $3x_1 + 3x_2 = 6$.

So we can see that if we want to maximize the objective function, we want to draw lines that are parallel to those two blue lines in Figures 2a and 2b that go as far as possible up. The figure 2d shows the last parallel line that we can draw that intersects our feasible region at a point. Therefore, that gives us the optimal solution, which is the point in blue.

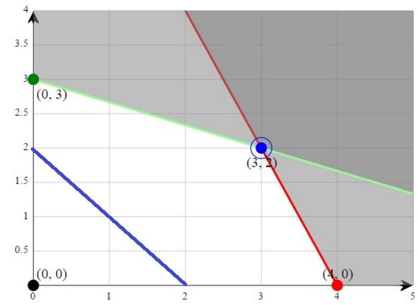
So the optimal solution is

$$(x_1, x_2) = (3, 2)$$

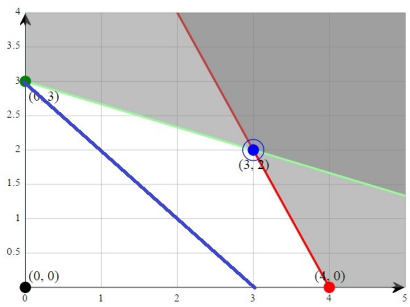
with optimal value 15.



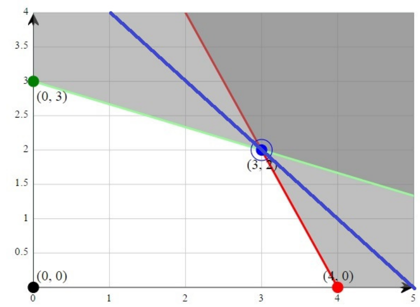
(a) Points satisfying $3x_1 + 3x_2 = 3$



(b) Points satisfying $3x_1 + 3x_2 = 6$



(c) Points satisfying $3x_1 + 3x_2 = 9$



(d) Points satisfying $3x_1 + 3x_2 = 15$

Figure 2: Different values of the objective function

Question 2

An oil refinery must blend two grades of gasoline to sell the resulting product. The first grade can be purchased from HyOctane, Inc. and has an octane number of 92. The second grade can be purchased from Allif Oil and has an octane number of 85.

The resulting blend must have an octane number of at least 89.

The refinery wants to produce exactly 120 barrels of the blend. They can purchase up to 90 barrels of each grade per week. The purchase price of the 92 octane grade is \$20 per barrel, and the 85 octane grade is \$15 per barrel.

The company must decide how many barrels of each grade should be used in the blend in order to minimize the production cost.

(a) What decisions must the company make?

Solution

How many barrels of the 85 octane and how many barrels of the 92 octane to be used in the blend.

(b) If we let

A= the number of barrels of 85 octane gasoline needed from Allif Oil for the blend

H= the number of barrels of 92 octane gasoline needed from HyOctane, Inc. for the blend

Then write an expression to represent the total gasoline purchase cost for the blend if he buys A barrels of gasoline from Allif Oil and H barrels of gasoline from HyOctane, Inc.

Solution

$20H + 15A$

(c) Write two inequality statements representing the gasoline purchase restrictions.

(d) Write the constraint that restricts the number of barrels of the blend that will be produced.

Solution

$A + H = 120$

(e) Now suppose you mix 5 barrel of Allif gasoline and 5 barrel of HyOctane gasoline. What is the average octane rating?

Solution

If we have 5 barrels of 85 octane and 5 barrels of 92 octane, then on average we have an 88.5 octane rating.

(f) Now suppose you mix 2 barrel of Allif gasoline and 6 barrel of HyOctane gasoline. What is the average octane rating?

Solution

Now here we are mixing more of the 92 octane gasoline and less of the 85 octane gasoline. So the octane rating should change accordingly.

The octane rating will be

$$\frac{2 \times 85 + 6 \times 92}{8} = 90.25$$

(g) Using the same idea, write a constraint that represents the average octane rating requirement.

Solution

The average octane rating will be:

$$\frac{A \times 85 + H \times 92}{A + H}$$

Therefore, we can rewrite the constraint as:

$$\frac{A \times 85 + H \times 92}{A + H} \geq 89$$

If we multiply both sides by $A + H$, we get

$$85A + 92H \geq 89A + 89H$$

or equivalently

$$3H \geq 4A$$

(h) Find the optimal solution to this problem.

Solution

As in the previous problem, we will graph the feasible region. The constraint $A + H = 120$ is equivalent to $H = 120 - A$ and is drawn in light green.

Now, we write $3H \geq 4A$ as $H \geq 4/3A$ and it is drawn in dark green. We are interested in the are above the dark green line and below the purple line (constraint $H \leq 90$).

Since all that region lies in the region $A \leq 90$, we are in good shape.

Now all we need is to draw the level curves of the objective function.

These are shown in figure 4.

From Figures 4a and 4b we see that we want to move the blue line as far as possible down while still intersecting the feasible region. That is achieved in Figure 4c with the minimizer (optimal solution) being at the interception of the equations $A + H = 120$ and $3H = 4A$.

The optimal solution is therefore $(A, H) = (\frac{360}{7}, \frac{480}{7})$ with optimal value 15000/7.

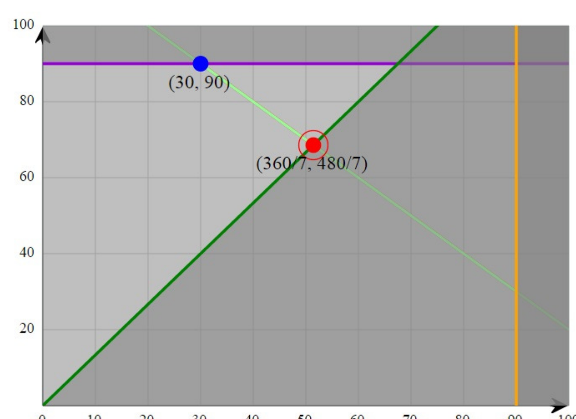
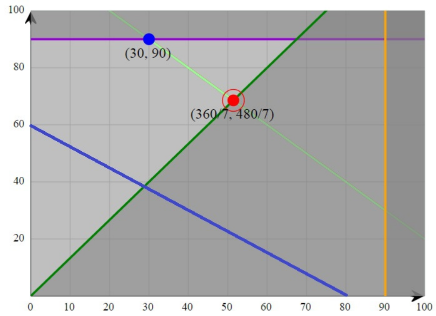
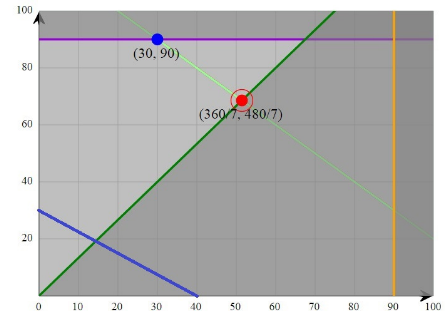


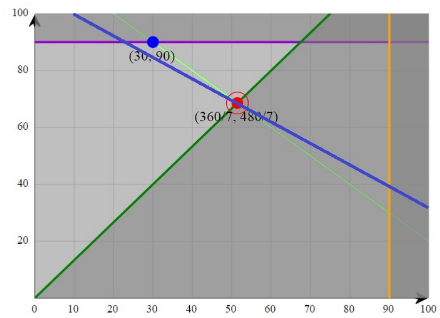
Figure 3: Feasible region



(a) Points satisfying $15A + 20H = 1200$



(b) Points satisfying $15A + 20H = 600$



(c) Points satisfying $15A + 20H = 15000/7$

Figure 4: Different values of the objective function

Question 3

As machines grow older, their repair and maintenance costs increase. However, replacing them requires capital investment. An important issue here is how often to buy and replace machines. Assume that we have a 6 year planning horizon.

Consider the cost of purchasing the machines in years 1 through 6 as follows:

	Year					
	1	2	3	4	5	6
Purchase cost	20	19	16	21	18	22

Also, consider that, as machines get older, they require more maintenance, and thus their annual repair and maintenance costs increase. Consider the following repair and maintenance costs based on the age of the machine as follows:

	Age					
	0	1	2	3	4	5
Repair and maintenance cost	1	3	4	6	7	8

How can one determine the best repair policy using the shortest path problem?

Question 4

Farmer Jones must determine how many acres of corn and wheat to plant this year. An acre of wheat yields 25 bushels of wheat and requires 10 hours of labor per week. An acre of corn yields 10 bushels of corn and requires 4 hours of labor per week. All wheat can be sold at $4 a bushel and all corn can be sold at $3 a bushel. Seven acres of land and 40 hours per week of labor are available. Government regulations require that at least 30 bushels of corn be produced during the current year. He wants to plant corn and wheat to maximize his total revenue.

Let x_1 be the number of acres of corn planted, and x_2 be the number of acres of wheat planted. Using these decision variables, answer:

(a) Is $x_1 = 2$, $x_2 = 3$ in the feasible region? What is its objective function value?

Solution

It is not in the feasible region since it violates the minimum number of bushels produced.

Its objective value is 360

(b) Is $x_1 = 4$, $x_2 = 3$ in the feasible region? What is its objective function value?

Solution

No it is not. It violates the maximum labor restriction.

Its objective value is 420

(c) Is $x_1 = 2$, $x_2 = -1$ in the feasible region? What is its objective function value?

Solution

No, since it violates nonnegativity. Its objective value is -70.

(d) Is $x_1 = 3$, $x_2 = 2$ in the feasible region? What is its objective function value?

Solution

Yes it is.

Its objective value is 260.

(e) What is the objective function?

Solution

Note that 1 acre of corn planted yields 25 bushels of wheat, which are sold by $4 a bushel. Therefore, each acre planted gives us $100.

Similarly, each acre of corn yields 10 bushels, which are sold at $3 a bushel, so each acre planted gives us $30.

The objective function is $30x_1 + 100x_3$

(f) What are the constraints?

Solution

Constraint representing total acres planted:

$$x_1 + x_2 \leq 7$$

Constraint representing hours of labor.

$$4x_1 + 10x_2 \leq 40$$

Minimum number of bushels of corn produced

$$10x_1 \geq 30$$

Nonnegativity constraints:

$$x_1 \geq 0$$
$$x_2 \geq 0$$

(g) Without finding the optimal solution to this problem, solve the following two questions (* marks a challenge question).

Give a lower bound on the optimal value, i.e., give a number that you can guarantee that the optimal solution will have value greater than or equal to that number. What is the largest lower bound that you can get?

Solution

Answers will vary.

Any feasible solution will give a lower bound on the optimal value. For instance, from above, we get a lower bound of 260.

(h) * Give an upper bound on the optimal value, i.e., give a number that you can guarantee that the optimal solution will have value less than or equal to that number. What is the smallest upper bound that you can get?

Solution

Answers will vary.

One possible bound can be derived as follows. We can see that we can plant at most 7 acres of anything and the most money we get per acre is 100, so we can get at most $700.

(i) Find the optimal solution to this problem.

Solution

Similarly to what was done before, we draw the feasible region (shown in white in Figure 5) and draw the level lines for the objective, finding the optimal solution to be $(x_1, x_2) = (3, 14/5)$ at the intersection of $x_1 \geq 3$ and $4x_1 + 10x_2 \leq 40$.

Figure 5: Feasible region

Question 5

Gemstone Tool Company (GTC) is a privately held firm in the consumer and industrial market for construction tools. The Winnipeg plant only produces wrenches and pliers. Wrenches and pliers are made from steel, and the production process involves molding the tools on a molding machine and then assembling the tools on an assembly machine. To make one wrench requires 1.5 lbs of steel, 1 hour on the molding machine and 0.3 hours on the assembly machine. To make one plier requires 1 lb of steel, 1 hour on the molding machine and 0.5 hours on the assembly machine. There are only 27 lbs of steel, 21 hours of the molding machine and 21 hours of the assembly machine available on any particular day. Wrenches are sold for \$1.30 and pliers are sold for \$1.00.

(a) What are the decisions that GTC must take? (decision variables)

Solution

How many wrenches and pliers to produce.

(b) What is the objective function?

Solution

If we let x_1 be the number of wrenches produced and x_2 the number of pliers, then the objective function is

$$1.30x_1 + x_2$$

(c) What are the constraints?

Solution

Maximum steel constraint:

$$1.5x_1 + x_2 \leq 27$$

Maximum molding constraint:

$$1x_1 + x_2 \leq 21$$

Maximum assembly constraint:

$$0.3x_1 + 0.5x_2 \leq 21$$

Nonnegativity constraints:

$$x_1 \geq 0$$

$$x_2 \geq 0$$

(d) Find the optimal solution to this problem.

Solution

Similarly to what was done before, we draw the feasible region (shown in white in Figure 6) and draw the level lines for the objective, finding the optimal solution to be $(x_1, x_2) = (12, 9)$ at the intersection of

$$1.5x_1 + x_2 \leq 27$$

and

$$1x_1 + x_2 \leq 21$$

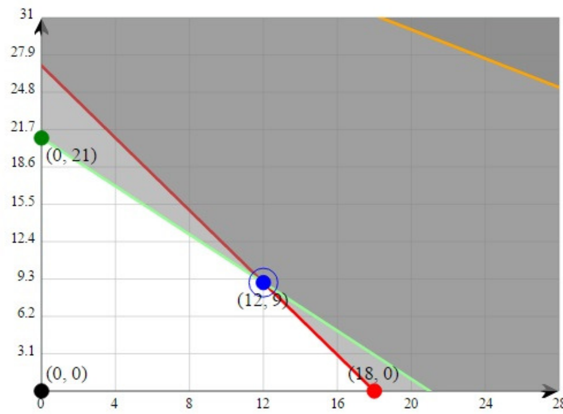
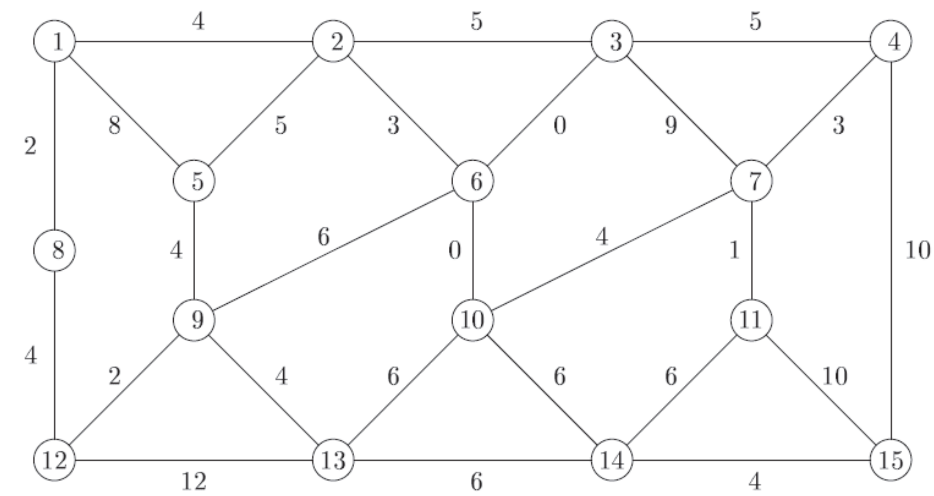


Figure 6: Feasible region

Question 6

Consider the following graph:



Answer the following questions without using the shortest path algorithm (* represents a challenge question)

(a) Find a path from vertex 1 to vertex 15.
Solution
Answers will vary.
One possible path is (1,2,3,4,15)

(b) What is the total cost of this path?
Solution
Answers will vary.
For the path above, the cost is 24

(c) What is the path of least total cost that you can find from vertex 1 to vertex 15?
Solution
Answers will vary.
One short path I could find without running any algorithm is
(1,2,6,10,14,15)

(d) How many different paths can you find from vertex 1 to vertex 15?
Solution
There is a huge number of paths (646 in total). Here are a few examples.
(1,2,3,4,7,10,6,9,12,13,14,15)
(1,2,3,4,15)

(e) Give an upper bound on the cost of the shortest path, i.e., give a number that you can guarantee that the shortest path will have value less than or equal to that number. What is the smallest upper bound that you can get?
Solution
Answers will vary. But any feasible solution gives an upper bound on the cost of the shortest path.

(f) * Give a lower bound on the cost of the shortest path, i.e., give a number that you can guarantee that the shortest path will have value greater than or equal to that number. What is the largest upper bound that you can get?
Solution
Answers will vary.
This will be explored in depth next time, but simple easy ones are: Picking the cheapest edge. Or finding that any path from 1 to 15 must have at least 4 edges and then picking the 4 cheapest edges, or can get more and more complex.

Now use the shortest path algorithm to find the actual shortest path from vertex 1 to vertex 15.

Solution

Here are the iterations of the shortest path algorithm (using $s = 1$):

Step	$v \in S$	$d(v)$	$s - v$-path	Adj. to S	$d'(v)$
1	1	0	(1)	2	4
				5	8
				8	2
2	8	2	(1, 8)	2	4
				5	8
				12	6
3	2	4	(1, 2)	3	9
				5	8
				6	7
				12	6
4	12	6	(1, 8, 12)	3	9
				5	8
				6	7
				9	8
				13	18
5	6	7	(1, 2, 6)	3	7
				5	8
				9	8
				10	7
				13	18
6	3	7	(1, 2, 6, 3)	4	12
				5	8
				7	16
				9	8
				10	7
				13	18
7	10	7	(1, 2, 6, 10)	4	12
				5	8
				7	11
				9	8
				13	13
				14	13
8	5	8	(1, 5)	4	12
				7	11
				9	8
				13	13
				14	13
9	9	8	(1, 8, 12, 9)	4	12
				7	11
				13	12
				14	13
10	7	11	(1, 2, 6, 10, 7)	4	12
				11	12
				13	12
				14	13
11	4	12	(1, 2, 6, 3, 4)	11	12
				13	12
				14	13
				15	22
12	11	12	(1, 2, 6, 10, 7, 11)	13	12
				14	13
				15	22
13	13	12	(1, 8, 12, 9, 13)	14	13
				15	22
14	14	13	(1, 2, 6, 10, 14)	15	17
15	15	17	(1, 2, 6, 10, 14, 15)		

So the shortest 1-15 path is (1,2,6,10,14,15) with cost 17.