Week 2 Problems - Robert Garbary, Fall 2014

- 1: Properties 1-3 of the norm in lecture notes, done during the talk. Go through the proofs I wrote down if you were confused when I explained them!

- 2: Show that Euclid's lemma is false if we drop the asumption that p is irreducible. More specifically, exhibit integers a, b, c where $a|bc$ but neither $a|b$ nor $a|c$.

- 3: Is 2 an irreducible element of $\mathbb{Z}[i]$? Is 3 irreducible?

- 4: (A bit harder) Let $\mathbb{Z}[\sqrt{-5}]$ denote the collection of all 'numbers' of the form

$$a + b\sqrt{-5}$$

with $a, b \in \mathbb{Z}$ (a,b integers). We can again add and multiply two of these numbers, and get another number of this form. For example the addition looks like

$$\begin{aligned} 1 + \sqrt{-5} + (2 - 4\sqrt{-5}) &= (1 + 2) + (1 - 4)\sqrt{-5} \\ &= 3 - 3\sqrt{-5} \end{aligned}$$

while the multiplication looks like

$$\begin{aligned} (1 + \sqrt{-5})(2 + 3\sqrt{-5}) &= 1 \bullet 2 + 1 \bullet 3\sqrt{-5} + \sqrt{-5} \bullet 2 + \sqrt{-5} \bullet 3\sqrt{-5} \\ &= 3 + 3\sqrt{-5} + 2\sqrt{-5} + 3(-5) \\ &= (3 - 15) + (3 + 2)\sqrt{-5} \\ &= -12 + 5\sqrt{-5} \end{aligned}$$

Prove that the only units of $\mathbb{Z}[\sqrt{-5}]$ are 1 and -1. That is, prove the following: let $x, y \in \mathbb{Z}[\sqrt{-5}]$ satisfy $xy = 1$. Prove that x and y are both ± 1.

- 5: Show that $\mathbb{Z}[\sqrt{-5}]$ does not have the unique factorization property by finding two different factorizations of 6.