



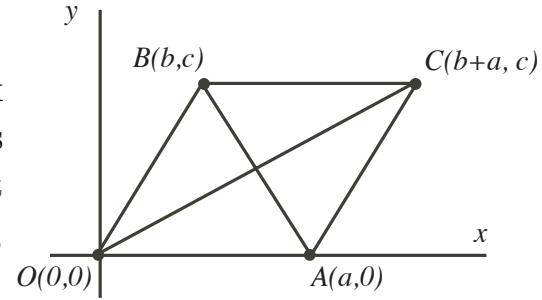
## Intermediate Math Circles

### Analytic Geometry II

#### Solutions to Problem Set April 1 2015

1. Prove, using analytic methods, that the diagonals of a parallelogram bisect each other.

**Proof:** Position the parallelogram with one vertex at  $O(0,0)$  and a second on the positive  $x$ -axis at  $A(a,0)$ ,  $a > 0$ . Position a third vertex at  $B(b,c)$ ,  $c \neq 0$ . Using properties of parallelograms, the fourth vertex would be  $C(b+a, c)$ .



To show that the diagonals bisect each other, we must show that the midpoints of the diagonals is the same point.

$$\text{midpoint } (OC) = \left( \frac{0 + (b+a)}{2}, \frac{0 + c}{2} \right) = \left( \frac{a+b}{2}, \frac{c}{2} \right)$$

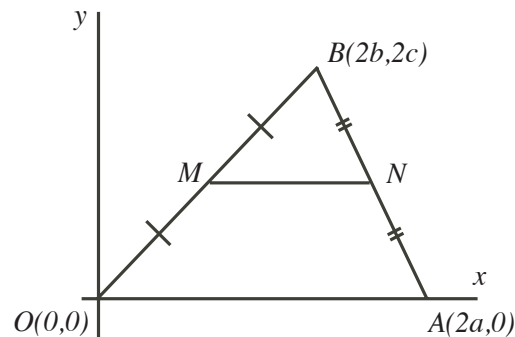
$$\text{midpoint } (AB) = \left( \frac{a+b}{2}, \frac{0+c}{2} \right) = \left( \frac{a+b}{2}, \frac{c}{2} \right)$$

Since  $\text{midpoint}(OC) = \text{midpoint}(AB)$ , then the midpoint is the same for both line segments and the diagonals bisect each other.



2. Prove, using analytic methods, that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and one-half the length of the third side.

**Proof:** Position the triangle with one vertex at  $O(0,0)$  and a second on the positive  $x$ -axis at  $A(2a,0)$ ,  $a > 0$ . Position a third vertex at  $B(2b,2c)$ ,  $c \neq 0$ . Let  $M$  be the midpoint of  $OB$  and  $N$  be the midpoint of  $AB$ . Using the formula for midpoint,  $M$  has coordinates  $(b,c)$  and  $N$  has coordinates  $(a+b,c)$ .



Since  $OA$  is along the  $x$ -axis,  $m(OA) = 0$ . The slope of  $MN = \frac{c-c}{a+b-b} = \frac{0}{a} = 0$ . Since  $m(OA) = m(MN) = 0$ , then  $OA \parallel MN$ .

It follows that  $MN$  is a horizontal line segment and its length is  $a + b - b = a$  units. But  $OA$  is a horizontal line segment with length  $2a$  units.  $\therefore MN = a = \frac{1}{2}(2a) = \frac{1}{2}(OA)$ .

Therefore, the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half of its length.

3. Determine the distance from the point  $Q(-3,5)$  to the line  $2x - 7y + 1 = 0$ .

Using  $d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$ ,  $A = 2$ ,  $B = -7$ ,  $C = 1$ ,  $x_1 = -3$ ,  $y_1 = 5$

$$\begin{aligned}
 d &= \frac{|2(-3) - 7(5) + 1|}{\sqrt{2^2 + (-7)^2}} \\
 &= \frac{|-6 - 35 + 1|}{\sqrt{4 + 49}} \\
 &= \frac{40}{\sqrt{53}} \\
 &= \frac{40\sqrt{53}}{53} \qquad \text{Multiply by } \frac{\sqrt{53}}{\sqrt{53}} \\
 &\doteq 5.5
 \end{aligned}$$

The distance from  $Q$  to the line is approximately 5.5 units.





4. Calculate the coordinates of the foot of the perpendicular from the point  $(2, -6)$  to the line  $x - 3y - 2 = 0$ .

**Solution:** Let  $P(a, b)$  be the foot of the perpendicular from  $Q(2, -6)$  to the line  $x - 3y - 2 = 0$ .

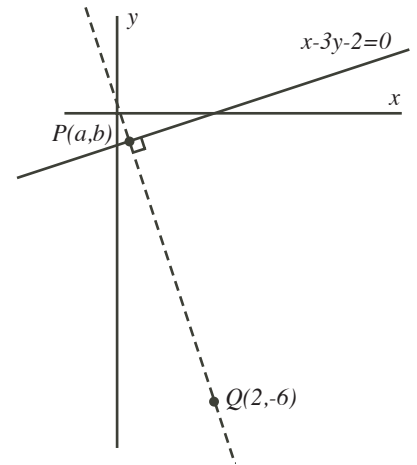
Rewrite the equation  $x - 3y - 2 = 0$  in slope-intercept form,  $y = \frac{1}{3}x - \frac{2}{3}$  (1).

The slope of the line is  $\frac{1}{3}$  so the slope of the perpendicular is  $-3$ . The perpendicular passes through  $Q(2, -6)$  so the equation is  $y - (-6) = -3(x - 2)$  or  $y + 6 = -3x + 6$  or  $y = -3x$  (2).

At the intersection,  $y = y$  so

$$\begin{aligned} -3x &= \frac{1}{3}x - \frac{2}{3} \\ -9x &= x - 2 \\ -10x &= -2 \\ x &= \frac{1}{5} \end{aligned}$$

Substituting in (2),  $y = -\frac{3}{5}$ . The coordinates of the foot of the perpendicular are  $\left(\frac{1}{5}, -\frac{3}{5}\right)$ .

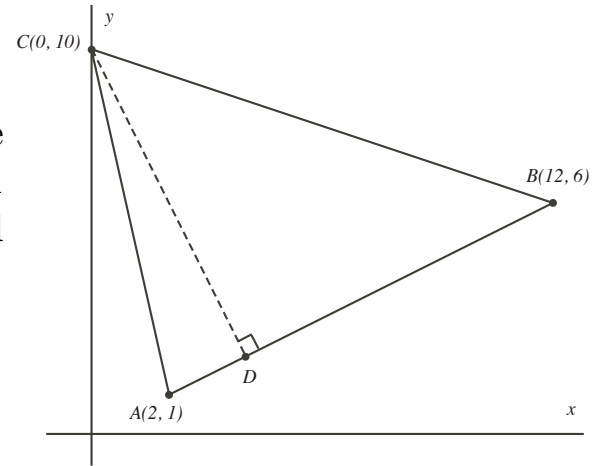




5. In  $\triangle ABC$ , with vertices  $A(2, 1)$ ,  $B(12, 6)$  and  $C(0, 10)$ , an altitude is drawn from  $C$  touching  $AB$  at  $D$ . Determine the length of the altitude  $CD$ . Develop two different solutions.

**Solution 1:** The slope of  $AB = \frac{6-1}{12-2} = \frac{1}{2}$ .

Using  $m = \frac{1}{2}$  and  $(2, 1)$ , the equation of the line through  $AB$  is  $y - 1 = \frac{1}{2}(x - 2)$  or  $y - 1 = \frac{1}{2}x - 1$  or  $y = \frac{1}{2}x$  or  $2y = x$  or  $x - 2y = 0$ , in standard form.



Using  $d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$ ,  $A = 1$ ,  $B = -2$ ,  $C = 0$ ,  $x_1 = 0$ ,  $y_1 = 10$

$$\begin{aligned} d &= \frac{|1(0) - 2(10) + 0|}{\sqrt{1^2 + (-2)^2}} \\ &= \frac{|0 - 20 + 0|}{\sqrt{1 + 4}} \\ &= \frac{20}{\sqrt{5}} \\ &= \frac{20\sqrt{5}}{5} \quad \text{Multiply by } \frac{\sqrt{5}}{\sqrt{5}} \\ &= 4\sqrt{5} \end{aligned}$$

The length of the altitude  $CD$  is  $4\sqrt{5}$  units.

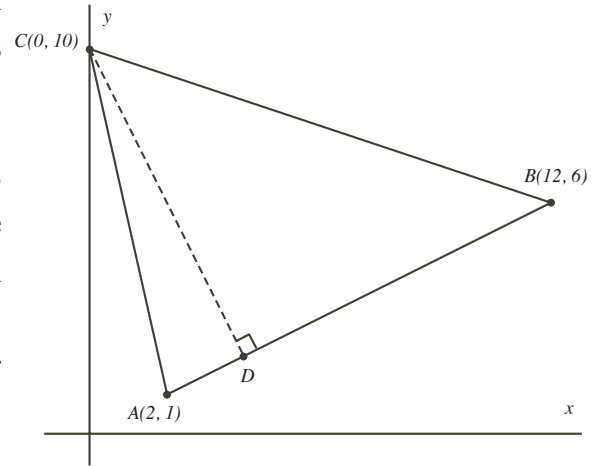




5. In  $\triangle ABC$ , with vertices  $A(2, 1)$ ,  $B(12, 6)$  and  $C(0, 10)$ , an altitude is drawn from  $C$  touching  $AB$  at  $D$ . Determine the length of the altitude  $CD$ . Develop two different solutions.

**Solution 2:** From solution 1, the equation of the line through  $AB$  is  $y = \frac{1}{2}x$  (1) and has slope  $m = \frac{1}{2}$ .

The altitude is perpendicular to (1) and has slope  $m = -2$  (since perpendicular slopes are negative reciprocals). It also passes through  $C(0, 10)$ . The equation of the line that contains the altitude is  $y - 10 = -2(x - 0)$  or  $y = -2x + 10$ . (2)



We can find the intersection of the two lines (1) and (2).

$$\begin{aligned} y &= y \\ \frac{1}{2}x &= -2x + 10 \\ x &= -4x + 20 \\ 5x &= 20 \\ x &= 4 \end{aligned}$$

Substituting  $x = 4$  into (1),  $y = \frac{1}{2}(4) = 2$ . The foot of the perpendicular is at  $D(4, 2)$ . We can now determine the length of  $CD$ .

$$\begin{aligned} CD^2 &= (0 - 4)^2 + (10 - 2)^2 \\ &= 16 + 64 \\ &= 80 \\ CD &= \sqrt{80}, \quad CD > 0 \\ &= \sqrt{16}\sqrt{5} \\ &= 4\sqrt{5} \end{aligned}$$

The length of the altitude  $CD$  is  $4\sqrt{5}$  units.





6. A point  $P$  is chosen on the line  $y = 2x + 3$  and a point  $Q$  is chosen on  $y = -x + 2$ . If the midpoint  $M$  of the line segment  $PQ$  is  $(2, 5)$ , determine the coordinates of  $P$  and  $Q$ .

**Solution:**

Let the  $x$ -coordinate of point  $P$  on  $y = 2x + 3$  be  $p$ . Thus, the  $y$ -coordinate of  $P$  is  $2p + 3$ . The point  $P$  is at  $(p, 2p + 3)$ .

Let the  $x$ -coordinate of point  $Q$  on  $y = -x + 2$  be  $q$ . Thus, the  $y$ -coordinate of  $Q$  is  $-q + 2$ . The point  $Q$  is at  $(q, -q + 2)$ .

$$\begin{aligned} \text{Midpoint } PQ &= (2, 5) \\ \left( \frac{p+q}{2}, \frac{(2p+3) + (-q+2)}{2} \right) &= (2, 5) \\ \left( \frac{p+q}{2}, \frac{2p-q+5}{2} \right) &= (2, 5) \end{aligned}$$

Since the two points are the same, the  $x$ -coordinates are equal and the  $y$ -coordinates are equal.

$$\begin{aligned} \frac{p+q}{2} &= 2 & \text{and} & & \frac{2p-q+5}{2} &= 5 \\ p+q &= 4 & \text{and} & & 2p-q+5 &= 10 \\ p+q &= 4 \quad (1) & \text{and} & & 2p-q &= 5 \quad (2) \end{aligned}$$

Adding equations (1) and (2), we obtain  $3p = 9$  and  $p = 3$  follows. Substituting  $p = 3$  in (1),  $q = 1$ .

Since  $P$  has coordinates  $(p, 2p + 3)$  and  $p = 3$ ,  $P$  is at  $(3, 9)$ .

Since  $Q$  has coordinates  $(q, -q + 2)$  and  $q = 1$ ,  $Q$  is at  $(1, 1)$ .

