Intermediate Math Circles
February 18, 2015
Patterns and Sequences

Here are the warmup problems to try as everyone arrives.

Fill in the blanks in the following patterns. (there may be more than one answer!)

1. $1, 2, 4, 8, \_\_\_, \_\_\_, \_\_\_$

2. $1, 8, 27, \_\_\_, \_\_\_$

3. $w, a, t, e, \_\_\_, \_\_\_, \_\_\_, \_\_\_$

4. $7, 8, 10, 13, 17, \_\_\_, \_\_\_, \_\_\_$

5. $3, \_\_\_, 4, \_\_\_, 5, \_\_\_, 6$

6. $1, 8, 11, 18, 80, \_\_\_, \_\_\_, \_\_\_$

Sequences

Initial definition: a list of numbers.

Asking a group of people to each contribute a number resulted in a sequence like:
$3, 5, 1, 27, 13, 29, 85, 9, \cdots$
Such a sequence can best be described as a random sequence. Although random sequences have a place in mathematics, they are not what we normally think about when we refer to a sequence. We also usually, and will here, restrict ourselves to sequences of Real numbers, avoiding Complex numbers.

When we think of a sequence of numbers, we think of it as having some order, or predictability, or rule, so that we can determine new terms (numbers) in the sequence.
Thus, a modified definition could be:
A sequence of real numbers is a list of real numbers which follow some rule for being part of the sequence.

How do we define a sequence?
We could give the first few terms and see if that is enough. For example. . .
2, 4, ...

The following were suggested as next terms.
6, the sequence is simply the even positive integers,
8, the sequence is defined by doubling a term to get the next term,
16, the sequence is defined by squaring a term to get the next term.
Thus we see that each of the above third numbers in the sequence make sense for a specific rule which defines the sequence.

Consider the sequence which begins

2, 4, 8, 16, . . .

What is the next term or number in the sequence? Many people would say 32.
If you were told that the next number(term) in the sequence is 3, can you determine the rule?

It is fairly clear that care must be taken to carefully define a sequence so that there can be no mistake about the terms of the sequence.

Notation.
t_1 is defined as the first term in a sequence.
t_2 is the second, t_3 is the third, t_{100} is the 100th and t_n is the nth or general term.

For example, the sequence made up of powers of 3 is $3, 9, 27, \cdots$.
We could say that $t_1 = 3$, $t_4 = 81$ and $t_n = 3^n$.

Examples

1. For the following sequences give the next 2 terms, t_{10} and t_n.
   (a) $3, 6, 9, 12.\_\_\_, \_\_\_, \cdots$ $\quad t_{10} = \_\_\_$ $\quad t_n = \_\_\_$
   (b) $2, 5, 8, 11.\_\_\_, \_\_\_, \cdots$ $\quad t_{10} = \_\_\_$ $\quad t_n = \_\_\_$
   (c) $2, 6, 12, 20.\_\_\_, \_\_\_, \cdots$ $\quad t_{10} = \_\_\_$ $\quad t_n = \_\_\_$
2. Given the general term for a sequence write down the first 4 terms of that sequence.
   (a) $t_n = 4n - 3$ $\quad \_\_\_, \_\_\_, \_\_\_, \_\_\_$
   (b) $t_n = n^3 + n^2 + 1$ $\quad \_\_\_, \_\_\_, \_\_\_, \_\_\_$
   (c) $t_n = \lfloor \frac{n^2}{3} \rfloor$ $\quad \_\_\_, \_\_\_, \_\_\_, \_\_\_$

Some special types of sequences.

Arithmetic Sequences
Arithmetic sequences are formed when the terms are formed by adding (or subtracting) a constant amount.
For example: $3, 11, 19, 27, \ldots$ or $100, 97, 94, 91, \cdots$

Geometric Sequences
Geometric sequences are formed when the terms are formed by multiplying (or dividing) by a constant amount.
For example: $2, 6, 18, 54, \ldots$ or $1210, 121, 12.1, 1.21, \cdots$

You will learn everything about these types of sequences in high school but for today we are going to do it by patterning rather than formulas!

Problem Set 1

1. Fill in the blanks in the following patterns. (there may be more than one answer!)

   (a) $0, 3, \_\_\_, 15, \_\_\_, \_\_\_, 48$

   (b) $\uparrow, \nearrow, \rightarrow, \searrow, \_\_\_, \_\_\_$

   (c) $2, 5, 7, 12, 19, 31\_\_\_, \_\_\_, \_\_\_$

   (d) $\frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{1}{4}, \_\_\_, \_\_\_, \_\_\_$

2. Identify the following sequences as Arithmetic, Geometric or Neither. Find t_{10} and t_n for each.

   (a) $3, 7, 11, 15, 19, \cdots$

   (b) $3, 33, 333, 3333, \cdots$

   (c) $2, 6, 18, 54, \cdots$

3. Find 5 well defined sequences that have first term 2 and have a later term 10.

Recursive Sequence Formula
The most famous Recursive Sequence is the Fibonacci Numbers or Fibonacci Sequence.
It is usually given as $1, 1, 2, 3, 5, 8, 13, \cdots$ but sometimes is given as $0, 1, 1, 2, 3, 5, 8, 13, \cdots$.
The formula is given by $t_1 = 1, t_2 = 1$ and $t_n = t_{n-1} + t_{n-2}$ for $n > 2$.

A recursive formula is rather ugly when you ask for a term like t_{100} because it requires you to find the first 99 terms to get term 100.
You can find a general term for some recursive formulas but not for all of them.

Examples

1. Write a recursive formula which describes $0, 2, 5, 9, 14, \cdots$

2. Write a recursive formula which describes $1, 3, 7, 15, 31, \cdots$

3. Write a general formula which describes $1, 3, 7, 15, 31, \cdots$

4. Write a recursive formula which describes $1, 1, 1, 3, 5, 9, 17, \cdots$

5. Write a recursive formula which describes $5, 10, 20, 40, 80, \cdots$

6. Write a general formula which describes $5, 10, 20, 40, 80, \cdots$

7. Use the formula $t_n = \frac{n(n+1)}{2}$ to find the first 5 terms of the sequence.

8. Use the formula $t_1 = 3, t_2 = 2$ and $t_n = t_{n-2} - t_{n-1}$ for $n > 2$ to find the first 8 terms of the sequence.

Problem Set 2

1. Starting at 9 and counting by 7's, a student counts 9, 16, 23, etc. What is the 15^{th} number the students says?

2. Find the general formula for the sequence defined by the recursive formula $t_1 = -3$ and $t_n = t_{n-1} + 5$ for $n > 1$.

3. Find the recursive formula for the sequence defined by the general formula $t_n = 2^n + 5$.

4. There are 23 bacteria cells in a laboratory dish at 1 pm. The bacteria triple in number every 15 minutes.

   (a) How many bacteria are there at 2:30 pm?

   (b) The clock in the lab chimes once at 1:00, twice at 2:00, three times at 3:00 and so on. What is the first number of chimes you will hear when there are at least 1 000 000 bacteria?

5. The numbers 2,5,8,11,14,$\cdots$ are written in order in a book, one hundred numbers to a page beginning on page one. The number 11 111 will be found on what page?

6. The numbers in the sequence 2, 7, 12, 17, 22, . . . increase by fives.
   The numbers in the sequence 3, 10, 17, 24, 31, . . . increase by sevens.
   The number 17 appears in both sequences. What is the next number which appears in both sequences?

7. In a sequence of six numbers, the first number is 4 and the last number is 47. Each of the numbers after the second is equal to the sum of the previous two numbers. Determine the sum of the six numbers.

8. The sum of the first n terms of a sequence is n(n + 1)(n + 2). Determine the 10th term of the sequence.