Intermediate Math Circles
February 25, 2015
Series and Induction

Here are the warmup problems to try as everyone arrives.

1. For the sequence $-7, -3, 1, 5, 9, \cdots$, find:

   (a) What kind of sequence is this?

   (b) t_{15}

   (c) t_n

   (d) Which term number is 401?

2. For the sequence $5, 15, 45, 135, \cdots$, find:

   (a) What kind of sequence is this?

   (b) t_n

   (c) t_{10}

3. Find the sum of $1 + 2 + 3 + 4 + \cdots + 999 + 1000$.

Series

A Series is the sum of the terms of a Sequence. There are well defined formulas for working with Arithmetic and Geometric Series that you will be taught but we are going to do everything using first principles which will allow us to look at non-Arithmentic/Geometric Series.

Remember Gauss.......

$$\begin{array}{rcrcrcrcccrcr}
\text{S} & = & 1 & + & 2 & + & 3 & + & \cdots & + & 99 & + & 100 \\
\text{S} & = & 100 & + & 99 & + & 98 & + & \cdots & + & 2 & + & 1 \\
\hline
2\text{S} & = & 101 & + & 101 & + & 101 & + & \cdots & + & 101 & + & 101 \\
2\text{S} & = & 100(101) & & & & & & & & & & \\
\text{S} & = & \frac{100(101)}{2} & & & & & & & & & & \\
\text{S} & = & 5050 & & & & & & & & & &
\end{array}$$

The same technique will work for any Arithmetic Series. (adding 1 to 100 is an Arithmetic Series!)

Example 1. Find the sum of $-7-3+1+5+9+\cdots+397+401$

Example 2. Find the sum of the first 50 multiples of 3.

Example 3. Find the sum of $101+99+97+\cdots+1-1-3-5$.

Geometric Series can be done with a different technique.

Example 1. Find the sum of $5 + 15 + 45 + 135 + \cdots + 295245 + 885735$

Example 2. Find the sum of $2048 - 1024 + 512 - \cdots + 2 - 1$.

Example 3. Find the sum of the first n terms of $3 + 6 + 12 + 24 + \cdots$.

We can use patterning to try to find the sum of Series that are not Arithmetic or Geometric.

Try making a chart to "guess" a formula for the sum of $1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2$.

Summation Notation

It is awkward to write all of the terms in a Series. We get around that by using the "$\cdots$" in the middle but this is rather long as well. We can make it nicer with Summation Notation sometimes known as Sigma Notation.

$$\sum_{n=1}^{10} n = 1 + 2 + 3 + \cdots + 9 + 10$$

1. Write the series that comes from each summation notation.

   (a) $\displaystyle\sum_{n=1}^{10} n^2 - n =$

   (b) $\displaystyle\sum_{n=3}^{17} 3n - 2 =$

2. Write the following series in Summation Notation.

   (a) $2 + 6 + 18 + 54 + \cdots + 2 \times 3^{65}$

   (b) $7 + 10 + 13 + \cdots + 97 + 100 =$

   (c) $2^3 + 3^3 + 4^3 + \cdots + 77^3 + 78^3 =$

The Principal of Mathematical Induction

Every statement in a sequence of statements $P_1, P_2, P_3, \cdots, P_n, \cdots$ is true if these steps are followed.

1. Verify the first statement P_1 is true.
2. Assume the kth statement, P_k is true, and use this to show that the next statement P_{k+1} is true.

Example 1. Prove that $(1)(3) + (2)(4) + (3)(5) + \cdots + (n)(n+2) = \frac{n(n+1)(2n+7)}{6}$.

Example 2. Prove that $\frac{n^3-n}{3}$ is an integer for every positive integer n.

Problem Set

1. Find the sum of the first 50 terms of the series $1 + 5 + 9 + 13 + \cdots$.
2. Find the sum of $6 + 9 + 12 + 15 + \cdots + 303 + 306$.
3. Find the sum of $2 + 10 + 50 + 250 + \cdots + 97656250$.
4. Write the series that comes from $\displaystyle\sum_{n=1}^{11} n^2 + 3$.
5. Write $1 + 8 + 27 + 64 + \cdots + 1000000$ in Summation Notation.
6. Prove that $1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.