Intermediate Math Circles
February 25, 2015
Solutions

Problem Set Solutions

1. Find the sum of the first 50 terms of the series $1 + 5 + 9 + 13 + \cdots$.

   The general term for the sequence is $t_n = 4n - 3$. Then $t_{49} = 193$ and $t_{50} = 197$.

$$\begin{array}{rclcccccccccc}
\text{S} & = & 1 & + & 5 & + & 9 & + & \cdots & + & 193 & + & 197 \\
\text{S} & = & 197 & + & 193 & + & 189 & + & \cdots & + & 5 & + & 1 \\
\hline
2\text{S} & = & 198 & + & 198 & + & 198 & + & \cdots & + & 198 & + & 198 \\
2\text{S} & = & 50(198) \\
\text{S} & = & \frac{50(198)}{2} \\
\text{S} & = & 4950
\end{array}$$

2. Find the sum of $6 + 9 + 12 + 15 + \cdots + 303 + 306$.

   The general term for this sequence is $t_n = 3n + 3$. Then 306 is the $101^t h$ term.

$$\begin{array}{rclcccccccccc}
\text{S} & = & 6 & + & 9 & + & 12 & + & \cdots & + & 303 & + & 306 \\
\text{S} & = & 306 & + & 303 & + & 300 & + & \cdots & + & 9 & + & 6 \\
\hline
2\text{S} & = & 312 & + & 312 & + & 312 & + & \cdots & + & 312 & + & 312 \\
2\text{S} & = & 101(312) \\
\text{S} & = & \frac{101(312)}{2} \\
\text{S} & = & 15756
\end{array}$$

   Alternate solution

   $S = 6 + 9 + 12 + \cdots + 303 + 306$
   $S = 3(2 + 3 + 4 + \cdots + 101 + 102)$
   $S = 3[(1 + 2 + 3 + \cdots + 101 + 102) - 1]$
   $S = 3[\frac{102(103)}{2} - 1]$
   $S = 15756$

3. Find the sum of $2 + 10 + 50 + 250 + \cdots + 97656250$.

$$\begin{array}{rclccccccccc}
\text{S} & = & 2 & + & 10 & + & 50 & + & \cdots & + & 97656250 & \\
5\text{S} & = & & & 10 & + & 50 & + & \cdots & + & 97656250 & + \; 488281250 \\
\hline
4\text{S} & = & \text{1488281250-2} \\
\text{S} & = & \frac{488281248}{4} \\
\text{S} & = & 122070312
\end{array}$$

4. Write the series that comes from $\displaystyle\sum_{n=1}^{11} n^2 + 3$.

   $4 + 7 + 12 + 19 + 28 + 39 + 52 + 67 + 84 + 103 + 124.$

5. Write $1 + 8 + 27 + 64 + \cdots + 1000000$ in Summation Notation.
   $\sum_{n=1}^{100} n^3$.

6. Prove that $1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
   Verify P_1
   $1^2 = \frac{1(1+1)(2\times1+1)}{6}$
   $1 = 1$

   Assume P_k
   $1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$

   Prove P_{k+1}
   $1^2 + 2^2 + 3^2 + 4^2 + \cdots + (k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$
   R.S.$= \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} = \frac{k(k+1)(2k+1)}{6}$

   L.S.$=1^2 + 2^2 + 3^2 + 4^2 + \cdots + (k+1)^2$
   $=[1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2] + (k+1)^2$
   $=[\frac{k(k+1)(2k+1)}{6}] + (k+1)^2$ We have used the assumption here!
   $=\frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$
   $=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$
   $=\frac{(k+1)[k(2k+1)+6(k+1)}{6}$
   $=\frac{(k+1)(2k^2+7k+6)}{6}$
   $=\frac{k(k+1)(2k+1)}{6}$
   =R.S.
   Therefore $1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ by induction.