# Intermediate Math Circles
# March 26, 2015
# Analytic Geometry I

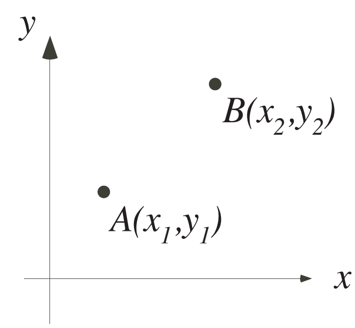
## 1. The Cartesian Plane

We use a coordinate system to allow us to translate a geometric problem into an algebraic problem.

We bring a lot to the table: angle properties and theorems, similar and congruent triangles, etc. In the first four weeks of the fall we examined Euclidean Geometry learning facts about angles, side lengths, and circles.

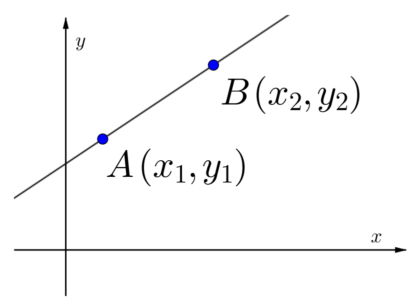
Credit for developments in the area of Analytic Geometry go to Rene Descartes. Descartes is also credited with the phrase: "*I think, therefore I am.*"

The coordinate system requires an origin, an $x$-axis, and a $y$-axis with which you should be familiar. Later in math we will extend to a third dimension.




A *point* is a specific location on the Cartesian Plane. The first diagram to the right shows two points $A$ and $B$.

A straight line can be drawn through two points. It has no beginning and no end. The second diagram to the right shows a line through the two points $A$ and $B$.




$AB$ is a line segment with endpoints $A$ and $B$. A line segment is different from a line because it has a fixed length. The third diagram shows line segment $AB$.

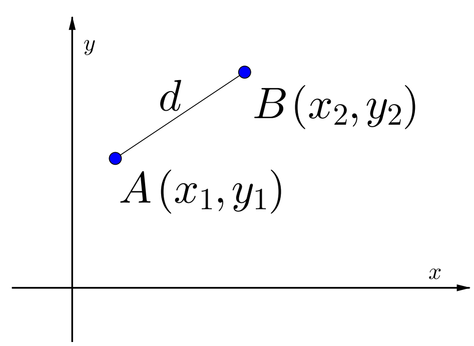
## 2. Distance Between Two Points

If $d$ is the distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ then

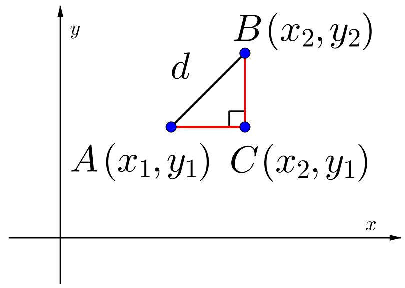
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

or

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$



## 2. Distance Between Two Points (continued)




If $d$ is the distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ then $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

**Proof:**

Form $\triangle ABC$ by drawing a vertical line segment from $B$ towards the $x$-axis and a horizontal line segment from $A$ intersecting the first line segment at $C$. The $x$-coordinate of $C$ will be the same as the $x$-coordinate of $B$ and the $y$-coordinate of $C$ will be the same as the $y$-coordinate of $A$. The coordinates of $C$ are $(x_2, y_1)$.
Since $\triangle ABC$ is right-angled,

$$d^2 = AB^2 = AC^2 + BC^2$$

But $AC$ is a horizontal distance and is easily calculated as $x_2 - x_1$ and $BC$ is a vertical distance and is easily calculated as $y_2 - y_1$.

$\therefore$, $d^2 = AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$. This is the second result shown earlier.

Taking the square root, $d = \pm\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

We can ignore $d = -\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ since the length $d \geq 0$.

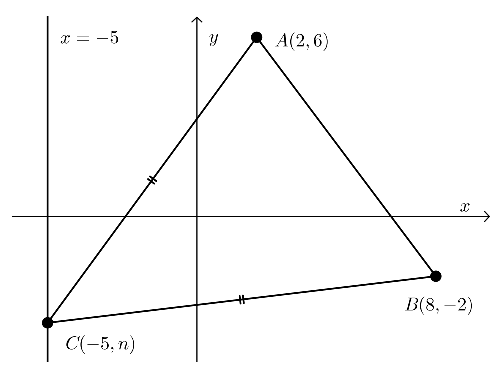
Our result, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ follows.

### Other Useful Information Concerning Distances

- The formula works with horizontal and vertical distances.
- The order of how the points are used in the formula does not matter.

### Problem (i)

The line segment joining $A(2, 6)$ to $B(8, -2)$ forms the base of isosceles $\triangle ABC$. The $x$-coordinate of the third vertex $C$ is $-5$ and $AC = BC$. Determine the $y$-coordinate of point $C$.

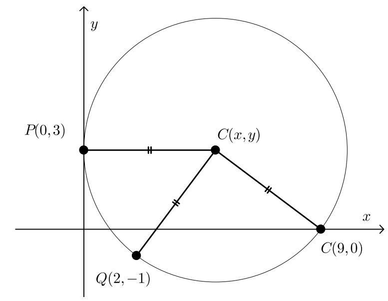



**Solution:**

Draw a good sketch to represent the given information. Let $C$ have coordinates $(-5, n)$. The point $C$ is somewhere on the vertical line $x = -5$.

$$\begin{aligned} \text{Since } AC &= BC \\ \text{It follows that } AC^2 &= BC^2 \\ (-5-2)^2 + (n-6)^2 &= (-5-8)^2 + (n+2)^2 \\ 49 + n^2 - 12n + 36 &= 169 + n^2 + 4n + 4 \\ -16n &= 88 \\ n &= -\frac{11}{2} \end{aligned}$$

The $y$-coordinate of $C$ is $-\frac{11}{2}$. This can be easily verified by calculating the distances with the points.




## Problem (ii)

Determine the centre of a circle which passes through points $P(0,3)$, $Q(2,-1)$, and $R(9,0)$.

## Solution:

Let $C(x,y)$ be the centre of the circle. Then $PC = QC = RC$ or $PC^2 = QC^2 = RC^2$.

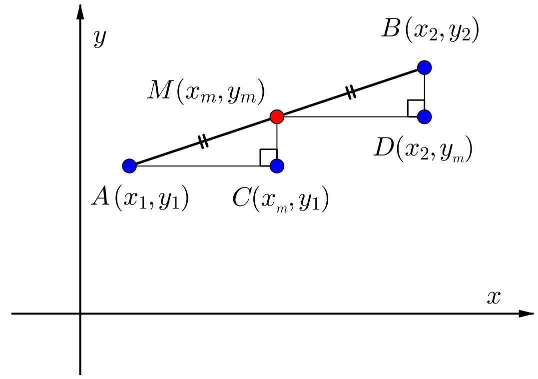
$$\begin{array}{llrcl}
\text{Since} & PC^2 = QC^2, & (x-0)^2 + (y-3)^2 & = & (x-2)^2 + (y+1)^2 \\
\text{Expanding} & & x^2 + y^2 - 6y + 9 & = & x^2 - 4x + 4 + y^2 + 2y + 1 \\
\text{Simplifying} & & -6y + 9 & = & -4x + 2y + 5 \\
 & & 4x - 8y & = & -4 \\
\text{Dividing both sides by 4} & & x - 2y & = & -1 \quad (1)
\end{array}$$

$$\begin{array}{llrcl}
\text{Since} & RC^2 = QC^2, & (x-9)^2 + (y-0)^2 & = & (x-2)^2 + (y+1)^2 \\
\text{Expanding} & & x^2 - 18x + 81 + y^2 & = & x^2 - 4x + 4 + y^2 + 2y + 1 \\
\text{Simplifying} & & -18x + 81 & = & -4x + 2y + 5 \\
 & & -14x - 2y & = & -76 \quad (2)
\end{array}$$

$$\begin{array}{lrcl}
\text{Using elimination, (1) - (2)} & 15x & = & 75 \\
 & x & = & 5 \\
 & \text{Substituting in (1) } 5 - 2y & = & -1
\end{array}$$

## 3. Midpoint Between Two Points

If $M(x_m, y_m)$ is the midpoint of $AB$, then $M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.




### Proof:

Draw a horizontal segment from $A$ intersecting a vertical segment from $M$ at $C$. The point $C$ will have the same $x$-coordinate as $M$ and the same $y$-coordinate at $A$. Therefore, $C$ is $(x_m, y_1)$. Similarly, draw a horizontal segment from $C$ intersecting a vertical segment from $B$ at $D$. The point $D$ will have the same $x$-coordinate as $B$ and the same $y$-coordinate at $M$. Therefore, $C$ is $(x_2, y_m)$.

The horizontal distance from $A$ to $C$ will equal the horizontal distance from $M$ to $D$. It follows that

$$x_m - x_1 = x_2 - x_m \Longrightarrow 2x_m = x_1 + x_2 \Longrightarrow x_m = \frac{x_1 + x_2}{2}$$

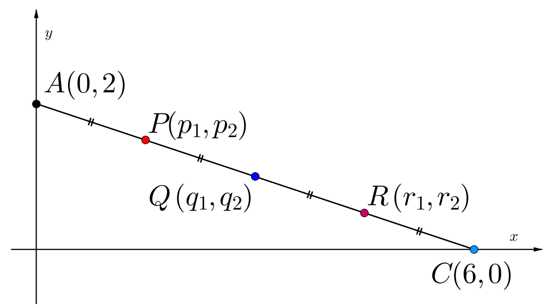
Similarly, the vertical distance from $C$ to $M$ will equal the vertical distance from $D$ to $B$. It follows that

$$y_m - y_1 = y_2 - y_m \Longrightarrow 2y_m = y_1 + y_2 \Longrightarrow y_m = \frac{y_1 + y_2}{2}$$

Therefore, $M$ has coordinates $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

### Problem (iii)

Points $P$, $Q$, and $R$ divide the line segment from $A(0, 2)$ to $C(6, 0)$, in that order, into four equal parts. Determine the coordinates of the three points, $P$, $Q$, and $R$.




Let $Q(q_1, q_2)$ be the midpoint of $AC$. Then, using the midpoint formula,

$$(q_1, q_2) = \left(\frac{0+6}{2}, \frac{2+0}{2}\right) = (3, 1)$$

The point $Q$ is $(3, 1)$.

Let $P(p_1, p_2)$ be the midpoint of $AQ$ since the segments are equal. Then, using the midpoint formula,

$$(p_1, p_2) = \left(\frac{0+3}{2}, \frac{2+1}{2}\right) = \left(\frac{3}{2}, \frac{3}{2}\right)$$

The point $P$ is $\left(\frac{3}{2}, \frac{3}{2}\right)$.

Let $R(r_1, r_2)$ be the midpoint of $QC$ since the segments are equal. Then, using the midpoint formula,

$$(r_1, r_2) = \left(\frac{3+6}{2}, \frac{1+0}{2}\right) = \left(\frac{9}{2}, \frac{1}{2}\right)$$

The point $R$ is $\left(\frac{9}{2}, \frac{1}{2}\right)$.

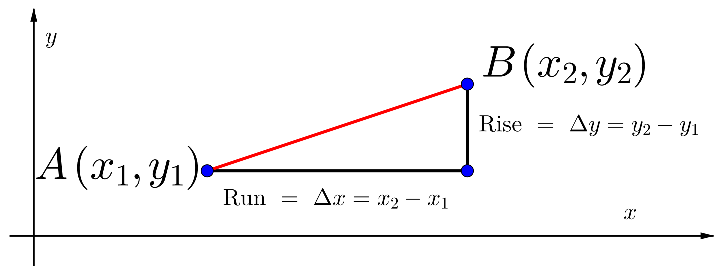
## 4. Slope of a Line / Line Segment

### (a) Definition

Slope is a measure of the steepness of a line (or line segment).

Slope is generally represented by the letter $m$.

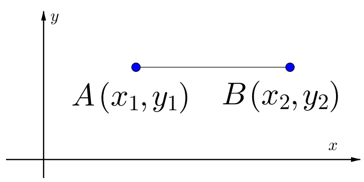
$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$




### (b) Special Cases

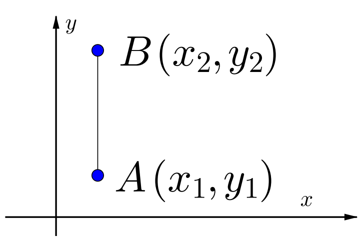
i) Horizontal Lines

The slope of a horizontal line is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0$.




ii) Vertical Lines

The slope of a vertical line is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{x_1 - x_1} = \frac{y_2 - y_1}{0}$. Division by zero is undefined. Therefore a vertical slope is undefined.



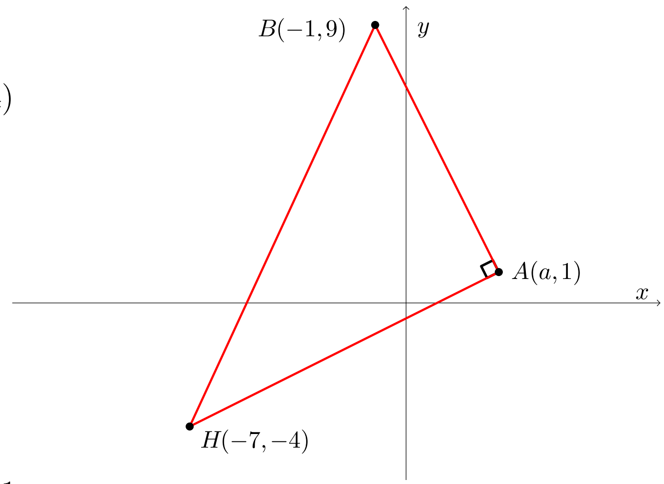

iii) Parallel Lines

If two lines are parallel, they have the same slope.

iv) Perpendicular Lines

If two lines are perpendicular, they have the negative reciprocal slopes. Also the product of the slopes of two perpendicular lines is $-1$.

## Problem (iv)




$\triangle BAH$ has vertices at $B(-1,9)$, $A(a,1)$ and $H(-7,-4)$ that $\angle BAH = 90°$. Determine the value of $a$.

Present two different solutions.

## Solution 1

Slope $BA = \frac{1-9}{a+1} = \frac{-8}{a+1}$.

Slope $HA = \frac{1+4}{a+7} = \frac{5}{a+7}$.

Since $\angle BAH = 90°$,

$$\begin{aligned} \text{slope}(BA) \times \text{slope}(HA) &= -1 \\ \frac{-8}{a+1} \times \frac{5}{a+7} &= -1 \\ \frac{-40}{(a+1)(a+7)} &= -1 \\ -40 &= -1(a+1)(a+7) \\ 40 &= a^2+8a+7 \\ 0 &= a^2+8a-33 \\ 0 &= (a-3)(a+11) \end{aligned}$$

$\therefore$, $a = 3$ or $a = -11$. There are two solutions. This may be surprising.

## Solution 2

$$\begin{aligned} BA^2 &= (a+1)^2 + (1-9)^2 = a^2+2a+1+64 = a^2+2a+65 \\ HA^2 &= (a+7)^2 + (1+4)^2 = a^2+14a+49+25 = a^2+14a+74 \\ BH^2 &= (-1+7)^2 + (9+4)^2 = 36+169 = 205 \end{aligned}$$

Since $\triangle BHA$ is right angled, using the Pythagorean Theorem,

$$\begin{aligned} BA^2 + HA^2 &= BH^2 \\ a^2+2a+65+a^2+14a+74 &= 205 \\ 2a^2+16a+139 &= 205 \\ 2a^2+16a-66 &= 0 \\ a^2+8a-33 &= 0 \\ (a-3)(a+11) &= 0 \end{aligned}$$

$\therefore$, $a = 3$ or $a = -11$.

Work on the problem set. Full solutions will be posted on the website later this week.