Math Circles: Population Models

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1 Examples

Now that you have seen the first slide set, let's work out some concrete examples. This will be followed by three problems for you to work out.

Example 1: The simplest growth problem Consider an island that has no goats until we introduce them. Assume that time is measured in years. At the beginning of the experiment, at year 0, we introduce 10 pairs of goats. After each year every two pairs of goats successfully produce one pair of kids (baby goats) that reach maturity. After each year one out of every ten pairs of goats dies off. Find the number of pairs of goats after ten years and after twenty years.

Let's let the population of *pairs* of goats be given by the variable P. Also let's let the subscript on P denote the year. So $P^{(0)}$ is the start of the experiment, and $P^{(10)}$ is the tenth year. We are told that

$$P^{(0)} = 10.$$

Next we need to figure out the population after 1 year. We write

$$P^{(1)} = P^{(0)} + \frac{1}{2}P^{(0)} - \frac{1}{10}P^{(0)}$$

and substitution gives

$$P^{(1)} = 10 + \frac{1}{2} \times 10 - \frac{1}{10} \times 10$$

so that

$$P^{(1)} = 14,$$

or fourteen pairs of goats.

So far so good. Now let's calculate the population after the second year

$$P^{(2)} = P^{(1)} + \frac{1}{2}P^{(1)} - \frac{1}{10}P^{(1)}$$

or

$$P^{(2)} = 14 + 7 - 1.4$$

and now we appear to be in trouble! The very sensible problem statement, which chose to specify population in terms of pairs of goats has given us the absurd prediction that 1.4 pairs of goats die. This is an example of a mismatch between a *mathematical* property, namely the nature of division for integers, and a desired property of the *model*. There is nothing wrong with the mathematics (in fact the whole being 'perfect' is the attraction of mathematics) but its application requires more care.

So how could we fix our model? Well perhaps the simplest possible way would be to make sure that the division operation gives an integer answer. One way to say that is 'ignore the remainder', another way is 'use the floor function' on the results of the division. The first way is more inline with the intuition you've developed in grade school about how mathematical operations are carried out. The second is more like what a computer might be programmed to do. The floor function is written like this

$$\lfloor 1.36 \rfloor = 1 \text{ and } \lfloor -1.36 \rfloor = -2.$$

You can see from the examples that for positive numbers the floor function is the same as the "ignore the remainder" of the division, but for negative numbers it is different. In fact, the proper way to define the floor function in words is to say that the floor of any number is the largest integer that is less than or equal to that number.

OK back to our example (it gets a new number just to remind you that it is different):

Example 2: The improved simplest growth problem Consider an island that has no goats until we introduce them. Assume that time is measured in years. At the beginning of the experiment, at year 0, we introduce 10 pairs of goats. After each year every two pairs of goats successfully produce one pair of kids (baby goats) that reach maturity. Mathematically this is expressed as the floor function applied to the result of dividing the number of pairs of goats by two. After each year one out of every ten pairs of goats dies off. Mathematically this is expressed as the floor

function applied to the result of dividing the number of pairs of goats by ten. Find the number of pairs of goats after five years.

Again let the population of *pairs* of goats be given by the variable P, and let the subscript on P denote the year. $P^{(0)}$ is the start of the experiment, and $P^{(10)}$ is the tenth year. We are told that

$$P^{(0)} = 10.$$

Next we need to figure out the population after 1 year. We write

$$P^{(1)} = P^{(0)} + \lfloor \frac{1}{2} P^{(0)} \rfloor - \lfloor \frac{1}{10} P^{(0)} \rfloor$$

and substitution gives

$$P^{(1)} = 10 + \lfloor \frac{1}{2} \times 10 \rfloor - \lfloor \frac{1}{10} \times 10 \rfloor$$

so that

$$P^{(1)} = 14,$$

or fourteen pairs of goats.

So far so good, and in fact the floor function has been unnecessary. Now let's calculate the population after the second year

$$P^{(2)} = P^{(1)} + \lfloor \frac{1}{2} P^{(1)} \rfloor - \lfloor \frac{1}{10} P^{(1)} \rfloor$$

or

$$P^{(2)} = 14 + 7 - 1 = 20.$$

Notice how the floor function made sure that the mathematics 'worked out'. This is useful, but a bit dangerous. For example, if I were to write a short computer program to calculate the result the program would finish almost instantly and I would have my 'answer'. But I would not really have seen the operations going into getting that answer, and thus I might not question whether my model is the a 'good' model (notice I did not say 'right' model, since no model is really 'right'). Ok, back to the calculation:

$$P^{(3)} = P^{(2)} + \lfloor \frac{1}{2} P^{(2)} \rfloor - \lfloor \frac{1}{10} P^{(2)} \rfloor$$

and I skip the in between steps to find
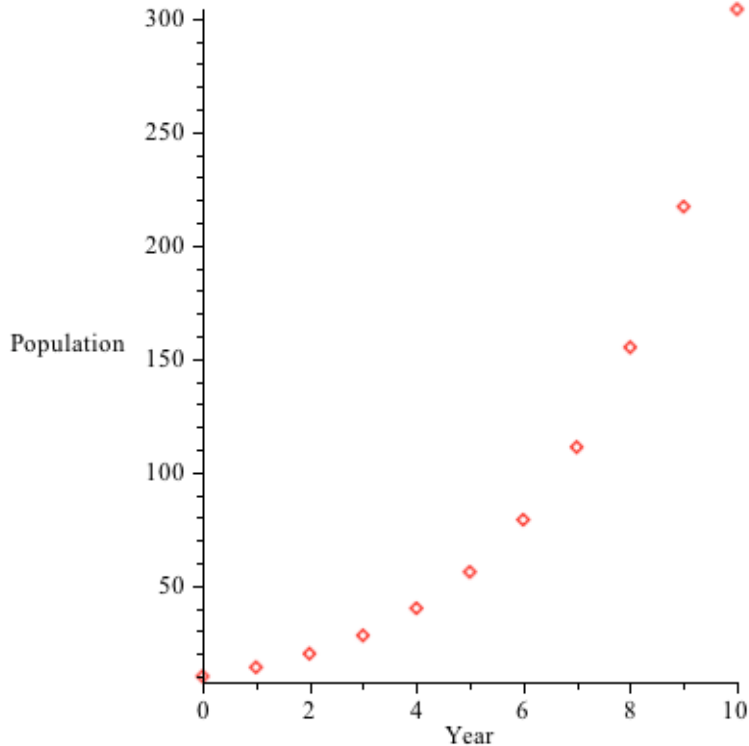
$$P^{(3)} = 28.$$

Figure 1.1: Population versus year for ten years.

An identical calculation gives, $P^{(4)} = 40$, and finally $P(5) = 56$.

When an instructor says 'I skipped steps' you should always be suspicious, and in this case I can willingly reveal that I lost interest in doing the calculation by hand and I wrote a one line program to help me do the calculation. Not only did this give me the answer I was looking for, but it allowed be to make a graph of the population after any number of years. Figure 1.1 shows what happens for the first 10 years.

Figure 1.2 shows what happens for the first 20 years.

The figures show the key feature of exponential growth of populations. In every day language we would say that the population growth increases as population grows. In nature this is clearly a bad idea, since the goats on the island would rapidly eat all the available food on the island. This should, in the long run affect the population of goats, but our model does not take this into account. Incidentally, the silly example of introducing goats onto an island is actually what happened historically on islands in the Adriatic sea (between Italy and Croatia) and the plant life on many of the islands has been greatly changed by the voracious appetite of the goats!
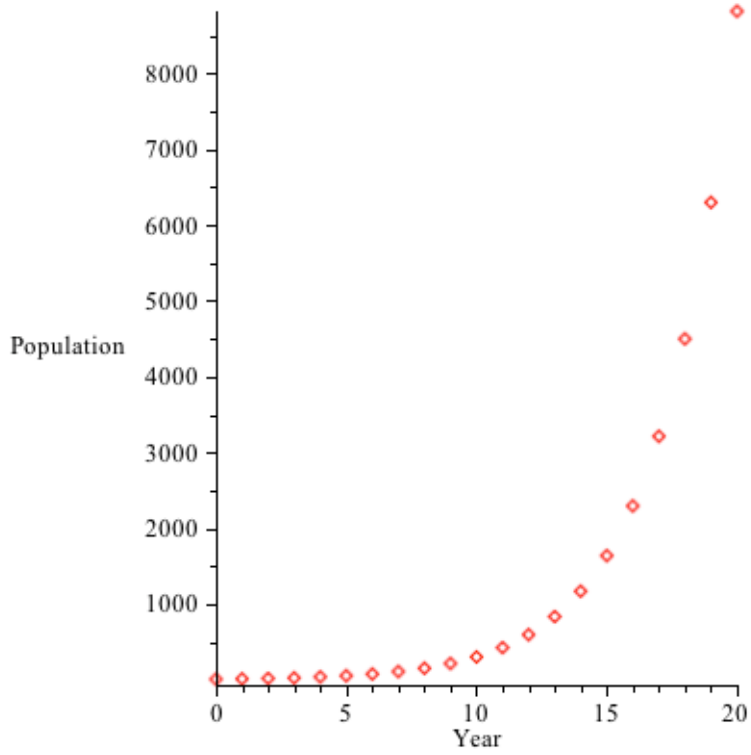
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Figure 1.2: Population versus year for twenty years.

2   Vocabulary

A scalar *recursion relation* is an equation through which we have a variable in the future specified in terms of the variable in the past. For example

$$P^{(n+1)} = 2P^{(n)},$$

or

$$P^{(n+1)} = 2P^{(n)} - P^{(n-1)}.$$

It is possible to have recursion relations between more than one variable. These are called *vector recursion relations* though we do not discuss them this week.

An *inhomogeneous recursion relation* is a recursion relation that has constant, or other terms that do not involve the changing variable. For example

$$P^{(n+1)} = 2P^{(n)} - 3,$$

or

$$P^{(n+1)} = 2P^{(n)} - 3n.$$

A *homogeneous recursion relation* does not have constant, or other terms that do not involve the changing variable. For example

$$P^{(n+1)} = 2P^{(n)},$$

and

$$P^{(n+1)} = 2P^{(n)} - P^{(n-1)}$$

are both homogeneous recursion relations.

An initial condition is a number used to start a recursion relation. For example with

$$P^{(n+1)} = 2P^{(n)},$$

$P^{(0)} = 3$ is a valid initial condition. Some recursion relations need more than one initial condition. For example

$$P^{(n+1)} = 2P^{(n)} - P^{(n-1)}.$$

would need two, so that $P^{(0)} = 3$, $P(1) = 8$ would work.

3 Problems

1. Suppose that a population of single celled organisms in a lab petri dish doubles in size every hour. Initially there are 100 organisms in the petri dish.

   (a) Write down a recursion relation for the number of organisms in the petri dish after n hours.

   (b) Solve the recursion relation to find a new expression for the population after n hours.

   (c) Use your model to predict how many organisms there will be after 4 hours.

   (d) Suppose additionally that 35 organisms are removed from the petri dish every hour. Write down a recursion relation to describe the population after n hours.

   (e) Solve the recursion relation to find a new expression for the population after n hours.

   (f) Use your new model to predict how many organisms there will be after 4 hours.

2. Suppose that a mature pair of rabbits can give birth to a new pair of rabbits every month. Additionally, it takes a new pair of rabbits two months to reach maturity before they are able to reproduce. Initially there is one pair of immature rabbits.

   (a) Write down a recursion relation for the number of pairs of rabbits after n months.

   (b) Solve the recursion relation to find an expression for the number of pairs of rabbits after n months.

   (c) How many rabbits will there be after one year? (Use your calculator for this one!)

3. A large aquarium tank has room for 100 fish to survive. Suppose that a particular species of aquarium fish reproduces every month at a rate that is twice the *percentage* of space that is left in the tank. Initially there are 7 fish in the aquarium.

   (a) Write down a recursion relation for the number of fish in the tank after n months.

   (b) Calculate the population of fish in the tank every month for the first 5 months. Describe what happens to the fish population in the tank.

   (c) How is this population model different from the ones we have considered above?

4 Extensions

In the second set of notes we showed you the so–called logistic model. Here we we ask you to play around with some of its properties. The recursion relation for this example is given by

$$P^{(n+1)} = P^{(n)} + aP^{(n)}\left(1 - \frac{P^{(n)}}{P_c}\right)$$

i) Use a calculator to compare the case $P^{(0)} = 2$, $a = 1$ to the case $P^{(0)} = 2$, $a = 2$.

ii) We have used the floor function to ensure that the population is an integer. Propose a way to use the floor function on the logistic model and discuss how your modification affects the results in part i).

iii) Explain what happens when $P^{(n)}$ is much less than P_c. In particular relate the parameter a to the birth rate in the simple growth model we have discussed in the Examples.