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Grade 6 Math Circles

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Pythagorean Theorem Solutions

Triangles: They're Alright When They're All Right

Today, we're going to talk about a specific theorem that has to do with right-angled triangles. Triangles are arguably one of the most important fundamental shapes, besides circles. We're going to take a look at a proof of the theorem and we're going to apply the theorem in problems.

Warm-Up

Try to answer the following 4 questions in 2 minutes.

1. What is the length of the hypotenuse if the legs are 8 and 15? 17
2. What is the length of the missing leg if the hypotenuse is 25 and one leg is 7? 24
3. If the length of a rectangle is 16 cm, and diagonal is 34 cm, what is the width? 30 cm
4. Using Euclid's Formula where n is 36 and m is 75, find a Pythagorean Triple.
   $a = 4329$, $b = 5400$, $c = 6921$

Terminology

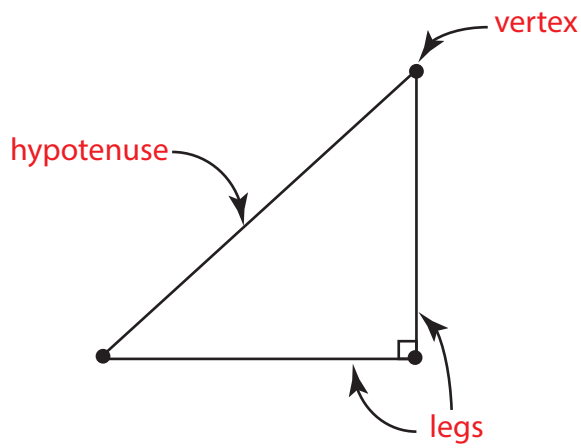
The Pythagorean Theorem can only be used in *right-angled triangles*. A right-angled triangle is a triangle with a 90° angle (a right angle).

The *hypotenuse* is the longest side of the right-angled triangle. It is opposite the right angle.

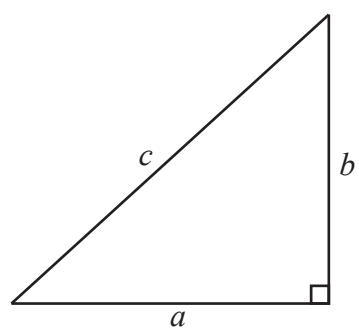
The *legs* of a triangle are the two other sides of a right-angled triangle.

A *vertex* is any corner of the triangle.

Try it out:
Label the diagram below.



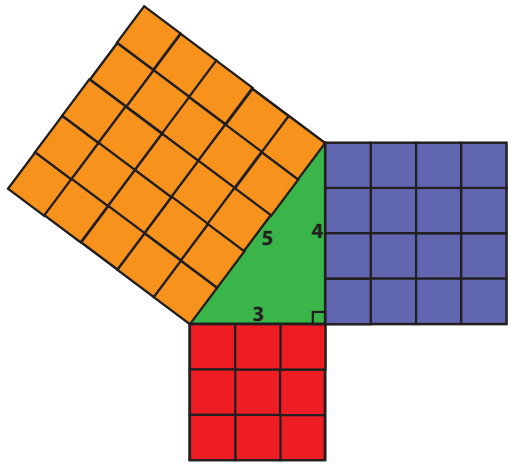
<u>The Theorem</u>
The Pythagorean Theorem states that $a^2 + b^2 = c^2$, where a and b are the legs and c is the hypotenuse.



Try it out:
Solve for the missing lengths.

1. $a = 1$, $b = \sqrt{3}$, $c =?$ 2
2. $a = 6$, $b =?$, $c = 10$ 8
3. $a =?$, $b = 24$, $c = 25$ 7

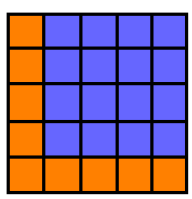
A Visualization of the Theorem



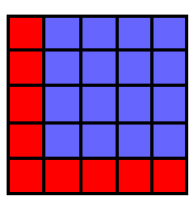
Examine the right-angled triangle above, with hypotenuse 5, and legs of 3 and 4. You can place squares on each of the sides. The hypotenuse has a 5×5 square, and the legs have 3×3 and 4×4 squares respectively.

This means that the hypotenuse has $5^2 = 25$ unit (1×1) squares, and the legs have $3^2 = 9$ and $4^2 = 16$ unit squares respectively.

Let's see if our two smaller squares fit into the bigger one:



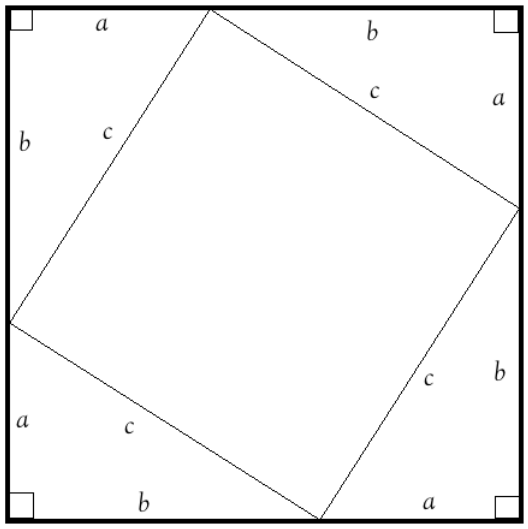
Now, it doesn't look like we could fit our 3×3 square in there; but notice that there are 9 orange 1×1 squares left, which is the same amount of red 1×1 squares that make up our 3×3 square! Let's break apart our red 3×3 square and fit it into the rest of our 5×5 square above.



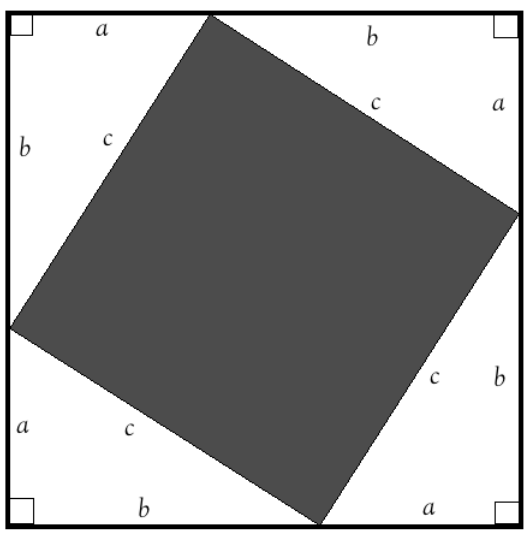
Well, it seems like the squares of the two smaller sides of the right triangle can be used to make up the square of the biggest side. In particular, $3^2 + 4^2 = 5^2$.

A Proof of the Theorem

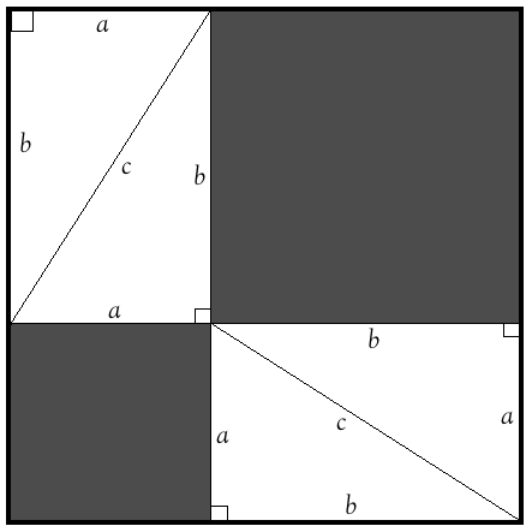
We will prove this theorem by using pictures. Let's arrange 4 of the same right triangles to get the figure below:



Well, if the 4 side lengths of the figure in the middle are all c, this tells us that we have a square. The shaded region below represents the area of the square in the middle of our figure. This area is equal to c^2.



Now, remember that we need to show that $a^2 + b^2 = c^2$. So, because we can, let's play around with the arrangement of the 4 right triangles in the previous figure:



Notice that the total area of our figure did not change, because all we did was move the right triangles around. Then, since the area of the 4 right triangles has also not changed, this tells us that the area of our two shaded regions must equal the area of our original shaded region, c^2. But, the area of the smaller shaded region is just a^2, and the area of the larger shaded region is just b^2.

Therefore, we conclude that $a^2 + b^2 = c^2$.

Pythagorean Triples

A Pythagorean Triple is a set of 3 numbers that form a perfect solution to the Pythagorean Theorem. The easiest triple to think of is

3 4 5

We can take any **multiple** of this triple (multiply all 3 numbers by the same number) and still have the Pythagorean Theorem work properly. Multiples of this triple are:

6 8 10

9 12 15

12 16 20

There are also other Pythagorean Triples that **are not** multiples of the triple 3 4 5.

5 12 13

7 24 25

9 40 41

11 60 61

Do you notice anything about these triples?

1. The 2nd two numbers only differ by 1.
2. The sum of the 2nd two numbers is equal to the square of the first number.

a	b	c	a^2	$\frac{a^2}{2}$
3	4	5	9	4.5
5	12	13	25	12.5
7	24	25	49	24.5
9	40	41	81	40.5
11	60	61	121	60.5

Try it out: Form a Pythagorean Triple with 13 as the first number. 13, 84, 85

Another way to find Pythagorean Triples directly is by using **Euclid's Formula**. The formula requires any two positive integers, m and n, where $m > n$.

$$a = m^2 - n^2$$
$$b = 2mn$$
$$c = m^2 + n^2$$

Proof:
$a^2 + b^2 = (m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2 = c^2$

Try it out: Form a Pythagorean Triple where m is 18 and n is 11. 203, 396, 445

Wrap-Up

Today, you looked at a very important theorem involving right-angled triangles. You will discover that the Pythagorean Theorem is extremely useful in a variety of physical situations and most geometry problems involving triangles.

Problem Set

Complete the following 15 problems. You may use a calculator.

1. Solve for the missing length, assuming c is the hypotenuse. Leave answers in radical form where necessary ($\sqrt{number}$).

   (a) $a = 2$, $b = 2$, $c =?$ $\sqrt{8}$

   (b) $a = 2$, $b = 4$, $c =?$ $\sqrt{20}$

   (c) $a = 84$, $b =?$, $c = 300$ 288

   (d) $a =?$, $b = 404$, $c = 505$ 303

   (e) $a = 28$, $b =?$, $c = 100$ 96

2. Find the Pythagorean Triple that contains a first number of 17. 17, 144, 145

3. Find the Pythagorean Triple where m is 72 and n is 14. 4988, 2016, 5380

4. Which of the following is NOT a Pythagorean Triple? (c)

   (a) 19 180 181

   (b) 35 612 613

   (c) 40 76 86

   (d) 12 35 37

5. Which of the following is a Pythagorean Triple? (d)

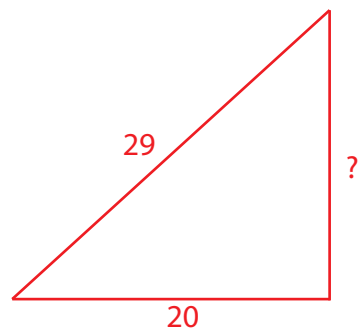
   (a) 21 24 33

   (b) 10 24 25

   (c) 3 3 6

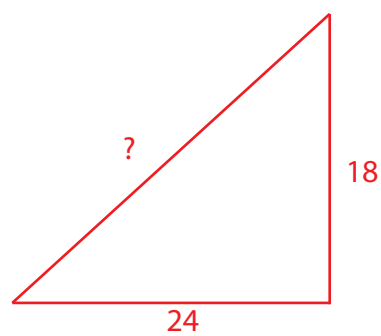
   (d) 22 120 122

6. If a bird is on the ground, 20 metres away from the base of a building, and the bird flies 29 metres to the top of the building, how tall is the building?
   Start by drawing a geometric representation of the situation:



   From here, it's a simple application of the Pythagorean Theorem. We find the height of the building is $\sqrt{29^2 - 20^2} = 21$ metres.

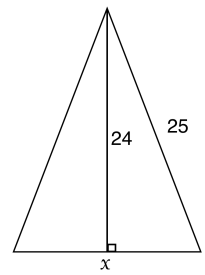
7. Harry is on his broom 18 metres above the point he took off from. Ron is standing on the ground, 24 metres away from Harry's takeoff point. How far apart are Harry and Ron?
   Start by drawing a geometric representation of the situation:



   From here, it's a simple application of the Pythagorean Theorem. We find that the distance between Ron and Harry is $\sqrt{18^2 + 24^2} = 30$ metres.

8. Solve for the missing lengths.

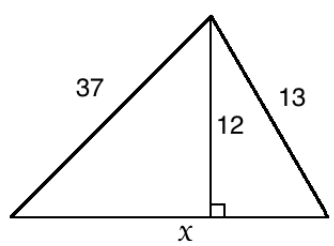
(a) Find x.



Using the lengths given, we can solve for $\frac{1}{2}$ the length of x. Therefore,

$$\begin{aligned} x &= 2\sqrt{25^2 - 24^2} \\ &= 2\sqrt{625 - 576} \\ &= 2\sqrt{49} \\ &= 2(7) \\ x &= 14 \end{aligned}$$

(b) Find x.



To solve for x, we'll solve for the length of the base of the two right triangles. For the base of the right triangle on the left (call it length a):

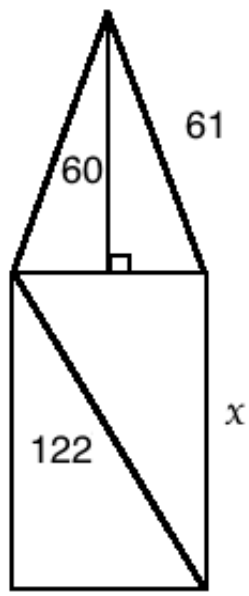
$$\begin{aligned} a &= \sqrt{37^2 - 12^2} \\ &= 35 \end{aligned}$$

For the right triangle on the right (call thing length b):

$$\begin{aligned} b &= \sqrt{13^2 - 12^2} \\ &= 5 \end{aligned}$$

Since $x = a + b$, this tells us that $x = 40$

(c) Find x.



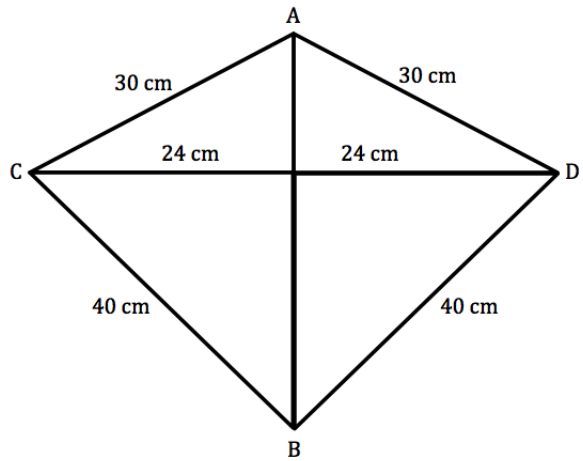
We'll first solve for the width of the rectangle. Call this w. Then,

$$\begin{aligned} w &= 2\sqrt{61^2 - 60^2} \\ &= 2\sqrt{3721 - 3600} \\ &= 2\sqrt{121} \\ &= 2(11) \\ w &= 22 \end{aligned}$$

Now we can solve for x using the Pythagorean Theorem again.

$$\begin{aligned} x &= \sqrt{122^2 - 22^2} \\ &= 120 \end{aligned}$$

9. Solve for the length of AB



For the top portion of AB (call this length x):

$$\begin{aligned} x &= \sqrt{30^2 - 24^2} \\ &= \sqrt{900 - 576} \\ &= \sqrt{324} \\ x &= 18 \end{aligned}$$

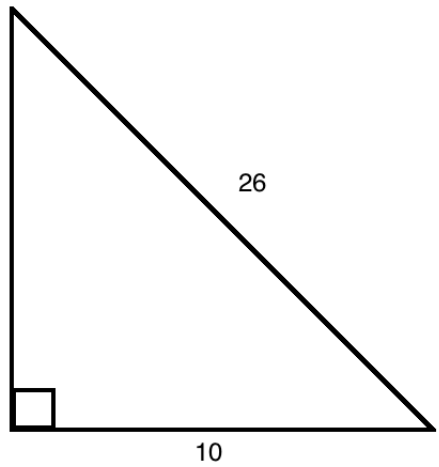
For the bottom portion of AB (call this length y):

$$\begin{aligned} y &= \sqrt{40^2 - 24^2} \\ &= \sqrt{1600 - 576} \\ &= \sqrt{1024} \\ y &= 32 \end{aligned}$$

Since the length of AB is just $x + y$, its length is 50 cm.

10. Solve for the area of the following shapes:

(a)



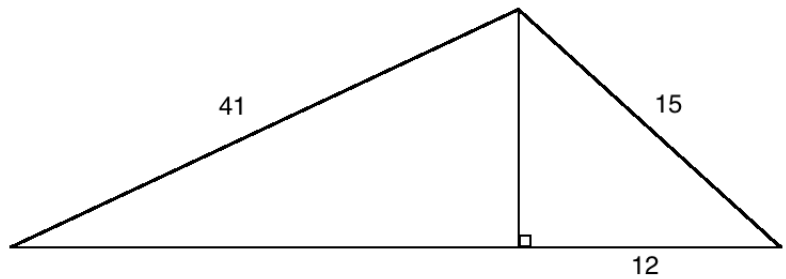
First we must solve for the height of this triangle. Call this h. Then,

$$\begin{aligned} h &= \sqrt{26^2 - 10^2} \\ &= \sqrt{676 - 100} \\ &= \sqrt{576} \\ h &= 24 \end{aligned}$$

Now for the area of the triangle:

$$\begin{aligned} A &= \frac{b \times h}{2} \\ &= \frac{10 \times 24}{2} \\ &= \frac{240}{2} \\ A &= 120 \end{aligned}$$

(b)



First we must solve for the height of this triangle. Call this h. Then,

$$\begin{aligned} h &= \sqrt{15^2 - 12^2} \\ &= \sqrt{225 - 144} \\ &= \sqrt{81} \\ h &= 9 \end{aligned}$$

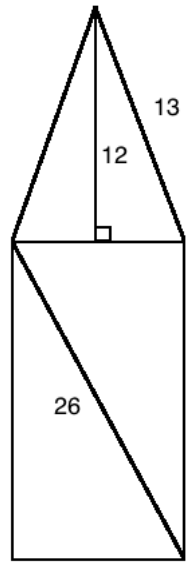
Now let's solve for the base of the right triangle on the left. Call this b_2. Then,

$$\begin{aligned} b_2 &= \sqrt{41^2 - 9^2} \\ &= \sqrt{1681 - 81} \\ &= \sqrt{1600} \\ b_2 &= 40 \end{aligned}$$

Then, combining our two bases we find the area of this triangle:

$$\begin{aligned} A &= \frac{b \times h}{2} \\ &= \frac{(40 + 12) \times 9}{2} \\ &= \frac{468}{2} \\ A &= 234 \end{aligned}$$

(c)



First we must solve for the width of the rectangle. Call this w. Then,

$$\begin{aligned} w &= 2\sqrt{13^2 - 12^2} \\ &= 2\sqrt{169 - 144} \\ &= 2\sqrt{25} \\ &= 2(5) \\ w &= 10 \end{aligned}$$

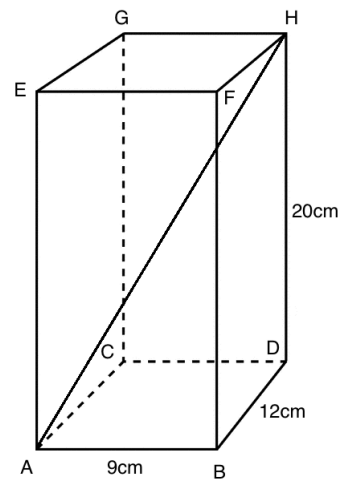
Note that the width of our rectangle is also the base of our triangle. Now we must solve for the length of our rectangle.

$$\begin{aligned} l &= \sqrt{26^2 - 10^2} \\ &= 24 \end{aligned}$$

The area of this figure is then

$$\begin{aligned} A &= \frac{10 \times 12}{2} + 10(24) \\ &= \frac{120}{2} + 240 \\ &= 60 + 240 \\ A &= 300 \end{aligned}$$

11. Given the rectangular prism below, solve for the length of AH.



To solve for AH, we will make a right triangle inside of the rectangular prism with hypotenuse AH and legs AD and DH. Since we know the length of DH (20 cm), we just need to solve for the length of AD. Call this length x. Then,

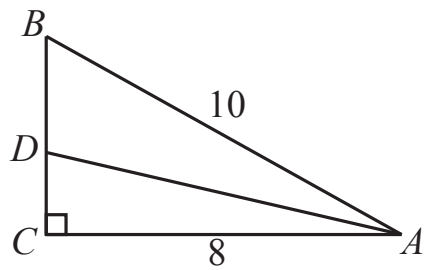
$$\begin{aligned} x &= \sqrt{9^2 + 12^2} \\ &= \sqrt{81 + 144} \\ &= \sqrt{225} \\ x &= 15 \end{aligned}$$

Now we can solve for the length of AH. Call this length y. Then,

$$\begin{aligned} y &= \sqrt{15^2 + 20^2} \\ &= \sqrt{225 + 400} \\ &= \sqrt{625} \\ y &= 25 \end{aligned}$$

Therefore, the length of AH is 25 cm.

12. Triangle ABC below is right-angled with $AB = 10$ and $AC = 8$. If $BC = 3DC$, what is the length of AD?



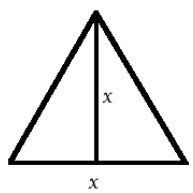
Using Pythagorean Theorem in triangle ABC, we obtain

$$8^2 + BC^2 = 10^2$$
$$BC^2 = 36$$
$$BC = 6$$

Since $BC = 3DC$, $DC = 2$. Using Pythagorean Theorem again in triangle ADC,

$$AD^2 = 2^2 + 8^2$$
$$AD^2 = 68$$
$$AD = \sqrt{68}$$

13. If the area of an isosceles triangle is 8 cm^2, and the base and height of the triangle are the same, what are the lengths of the other two sides?



Let x be the base and height of the isosceles triangle. Then, using the area, we can solve for x:

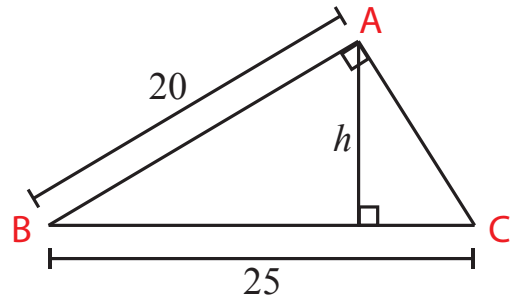
$$\begin{aligned} \frac{x \times x}{2} &= 8 \\ x^2 &= 16 \\ \sqrt{x^2} &= \sqrt{16} \\ x &= 4 \end{aligned}$$

Since this is an isosceles triangle, both bases of the two right triangles must be half the length of x (2 cm). We then solve for the hypotenuse of one of the right triangles, because this will give us the length of both (isosceles triangle). Call this length c. Then,

$$\begin{aligned} c &= \sqrt{2^2 + 4^2} \\ &= \sqrt{4 + 16} \\ c &= \sqrt{20} \end{aligned}$$

Therefore the length of the other two sides is $\sqrt{20}$ cm.

14. Calculate the value of h in the diagram below.



The area of the triangle can be determined from $\frac{AB \times AC}{2}$ since angle BAC is 90°. The area can also be determined from $\frac{BC \times h}{2}$. Then,

$$\frac{25 \times h}{2} = \frac{20 \times AC}{2}$$
$$5h = 4AC$$

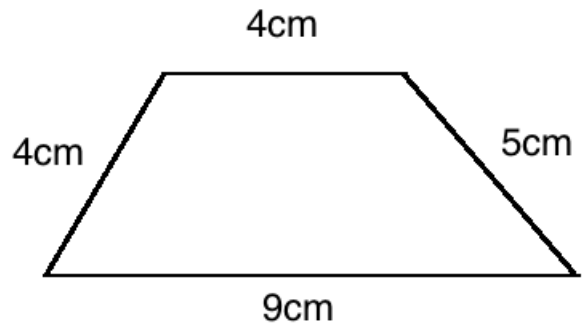
But since triangle ABC is right-angled,

$$\begin{aligned} AC^2 &= BC^2 - AB^2 \\ &= 25^2 - 20^2 \\ &= 225 \\ AC &= 15 \end{aligned}$$

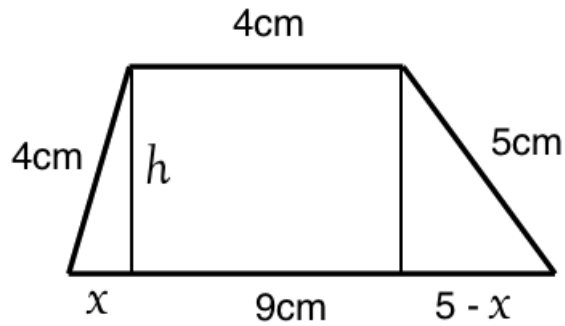
Then

$$5h = 4 \times 15$$
$$h = 12$$

15. Solve for the area of the trapezoid given the following information (round any calculations to one decimal place).



Let's make some right triangles within the trapezoid:



We cannot find the areas of these two triangles yet, but we can do some algebra to solve for the length of the base of both. We let x be the length of the base of the triangle on the left. This means that the length of the base of the triangle on the right is $5 - x$ (from $9 - 4 - x$). We'll now solve for these lengths with the fact that the heights of these two triangles are the same. We have

$$h^2 = 4^2 - x^2$$
$$h^2 = 5^2 - (5 - x)^2$$

So

$$\begin{aligned} 5^2 - (5 - x)^2 &= 4^2 - x^2 \\ 25 - (25 - 10x + x^2) &= 16 - x^2 \\ 25 - 25 + 10x - x^2 &= 16 - x^2 \\ 10x &= 16 \\ x &= 1.6 \end{aligned}$$

We can now solve for the height of the trapezoid. We'll use the triangle on the right:

$$\begin{aligned} h &= \sqrt{4^2 - 1.6^2} \\ &= \sqrt{16 - 2.56} \\ &= \sqrt{13.44} \\ &\approx 3.7 \end{aligned}$$

Now, for the area of the trapezoid, we'll find the sum of the areas of the two triangles and the rectangle in the middle:

$$A \approx \frac{1.6 \times 3.7}{2} + \frac{3.4 \times 3.7}{2} + 4(3.7) \tag{1}$$

$$\approx \frac{3.7(1.6 + 3.4)}{2} + 4(3.7) \tag{2}$$

$$\approx \frac{3.7(1.6 + 3.4 + 8(3.7)}{2} \tag{3}$$

$$\approx 3.7(\frac{5 + 8}{2}) \tag{4}$$

$$\approx 3.7(\frac{4 + 9}{2}) \tag{5}$$

$$\approx 24.1 \tag{6}$$

Notice in line 5 of our work above, we arrived at the formal equation for area of a trapezoid ($A = h(\frac{x+y}{2})$, where x and y are the lengths of the two bases). This is just showing we can find the area in more than one way after we have found the height of the trapezoid.