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Grade 7/8 Math Circles

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Pi Solutions

The Best Tasting Number

Today, we're going to talk about a very important number in mathematics. You will be fortunate enough to use this delicious-sounding number throughout the rest of your mathematical studies. This number has an interesting history and discovery, which we're going to talk about. We'll also take a look at the memorization of this fantastic number as well as a debate involving this number that has divided many mathematicians.

It's time to introduce you to your new favourite number, π.

Warm-Up

Try to answer the following 9 questions in 4 minutes without a calculator. The arithmetic tricks you learned last lesson may be useful. Don't worry if you can't answer them all. You'll be an expert by the end of this lesson!

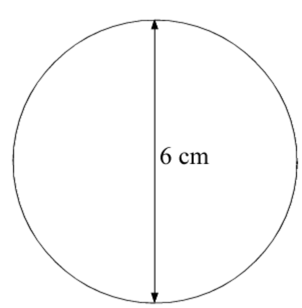
1. Write π to as many digits as you know. 3.14159...

2. Give a fractional approximation of π. Many answers, one would be $\frac{22}{7}$.

3. Express π as a ratio of the dimensions of a circle. $\pi = \frac{C}{d} = \frac{C}{2r} = \frac{A}{r^2}$

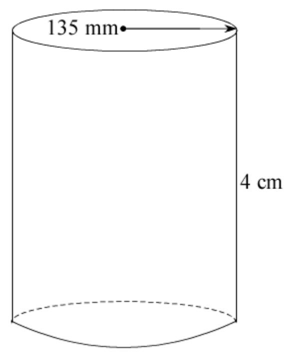
4. What is the *diameter* of the circle below (include units)? 6 cm

5. What is the *radius* of the circle below (include units)? 3 cm



6. What is the **exact** *circumference* of the circle on the previous page (include units)? 6π cm

7. What is the **exact** *area* of the circle on the previous page (include units)? 9π cm^2



8. What is the **exact** *volume* of the cylinder above (include units)? $7\,290\,000\pi$ mm^3 or equivalent. Note given units. Use trick from last week to find 135^2.

9. What is the **exact** *surface area* of the cylinder above (include units)? $144\,450\pi$ mm^2 or equivalent. Note given units.

The Life of Pi

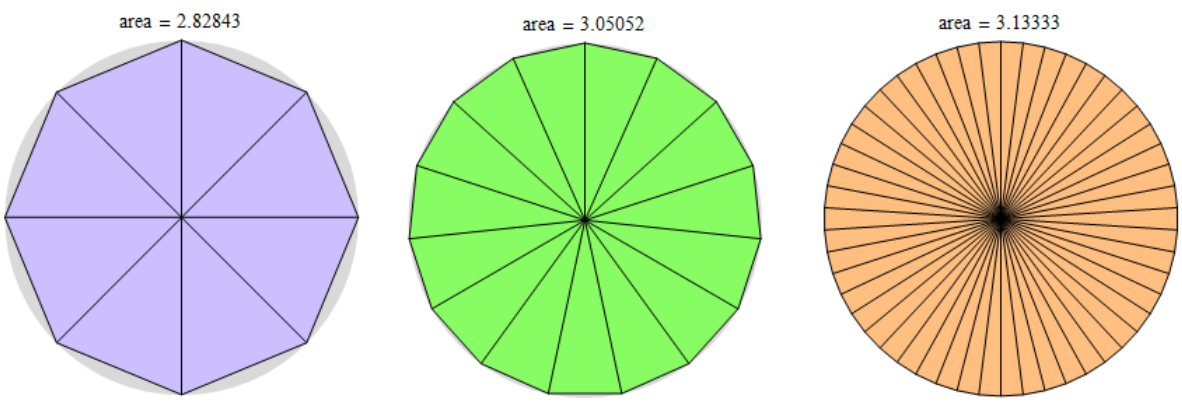
Warm-Up (WU) 1 and 2 tested your background knowledge of pi. Here are some fast facts about π:

- Pi is an irrational number.
  - An irrational number is a number that cannot be written as a ratio of two integers. It cannot be represented as a terminating or repeating decimal.
  - This result was proven in 1761 by Johann Lambert.
- Over 13 trillion (13 000 000 000 000) digits of pi are currently known
  - The world record for most digits of pi memorized is 67 890. You will most likely never need to know beyond the first six digits (3.14159).

Approximations of Pi

One of the oldest approximations of π was used by Babylonian mathematicians 40 centuries ago. They used the approximation $\frac{25}{8}$. Another approximation used around this time in history was $(\frac{16}{9})^2$. Many centuries later, Archimedes approximated circles with many-sided polygons to prove that $\frac{223}{71} < \pi < \frac{22}{7}$. The relationship of pi with a circle is fundamental, and we'll talk more in-depth about it later. One of the most common approximations of pi that is used is $\frac{22}{7}$.

Below, you will see a regular octagon, 15-gon, and 50-gon with areas calculated by the use of isosceles triangles. As you will notice, the area is approaching π. This is because the equal sides of the isosceles triangles have length 1 and as the number of sides in the polygon increases, the polygon's area approaches the area of a circle having radius 1.



Pictures and values generated by demonstration found at http://demonstrations.wolfram.com/ApproximatingPiWithInscribedPolygons/

In order to compute the areas using the triangles, you would need to utilize a branch of mathematics called trigonometry . This is the study of lengths and angles of triangles. We will look briefly into this area of mathematics next lesson. You will study trigonometry in detail in high school.

For the time being, here is a formula to calculate the area of a regular n-sided polygon using the isosceles triangle technique above. You can find the sin button on all scientific calculators. You will learn about sin in high school. Make sure the calculator is in degrees mode.

$$A = \frac{n}{2} \times \sin(180° - \frac{360°}{n})$$

Try it out:

Use the formula above and a calculator to find the area of a regular 96-gon (this is the shape Archimedes used, though with a different approach) and a regular 3000-gon.

96-gon: 3.139350; 3000-gon: 3.141590

There are now much more sophisticated, complicated, and exact formulas to compute pi. These range from *infinite series* to *continued fractions*. As you continue your career in the world of mathematics, you will learn more about these topics.

Piphilology

Some people love a good challenge. Memorizing digits of pi is a popular thing to do (amongst people who have a lot of time on their hands). Piphilology is the creation and use of special techniques to remember the digits of pi. One of the most popular techniques is through the use of "piems". These are poems that help you remember pi. This piem for example uses the number of letters per word to represent the digits of pi:

"Sir, I have a rhyme excelling,
In mystic power and magic spelling."

Try it out:

Make your own piem!

Why π?

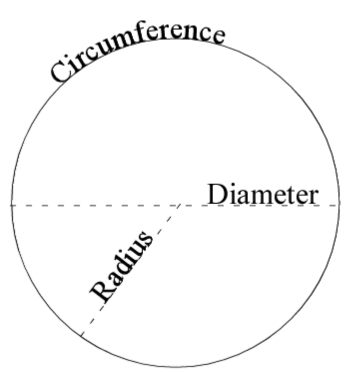
While it would be nice if pi was named because of its relationship to pie, which is of course in the shape of a circle, this is sadly not the case. The symbol π (or its uppercase version Π) is the Greek letter for 'p' (or 'P'). The symbol was first used in 1706 by British mathematician William Jones to represent the word periphery (perimeter). However, it was not adopted into common usage until the much more famous Leonhard Euler began using it in his works.

Circles and Pi

WU 3 through 7 tested your knowledge of the relationship between pi and circles. This is a fundamental concept to have in your mathematical toolkit.

Dimensions of a Circle

Below is an image showing the different dimensions of a circle. You should make sure to remember these.



The Fundamental Relationship

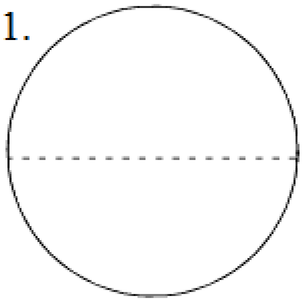
As a mathematician, it's important to learn how to discover concepts yourself. Math is not just memorization and formulas! You are now going to take a look into the world of mathematical experimentation.

Try it out:

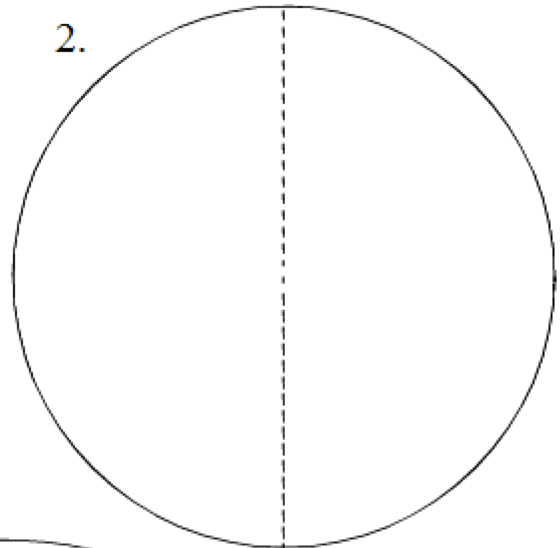
On the next page, you will find 3 circles of varying sizes, with dashed lines through their diameters. You are going to conduct an investigation into the dimensions of a circle.

1. Take some string and lay it along the diameter of circle 1. Cut it to the length of the diameter.
2. Starting at one end of the diameter line, wrap the string along the circumference of the circle. Make a tick mark at the point on the circumference where the string ends.
3. Keeping wrapping the string around the circumference starting at your tick mark (make a new mark every time the string ends) until you overlap with your starting point.
4. Estimate how many string lengths the circumference was. You may use a ruler to measure your string if you wish. Write your estimate on the line below the circle.
5. Repeat for circles 2 and 3.

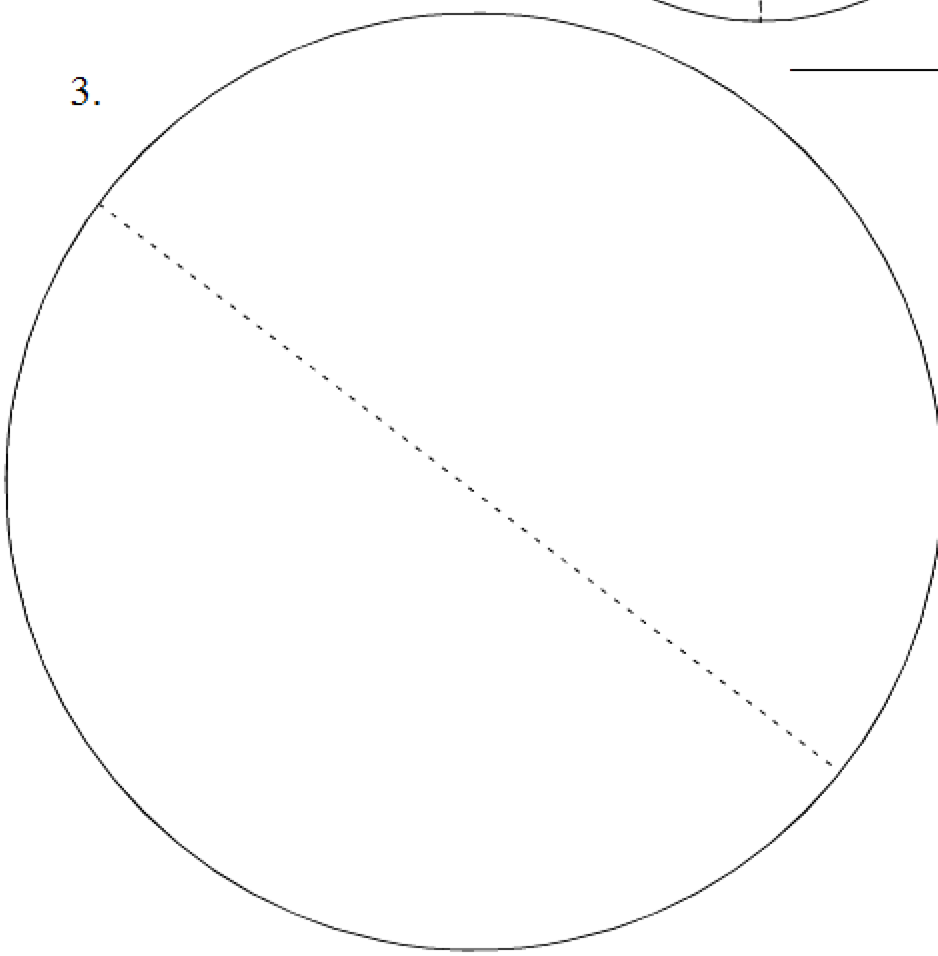
1.



2.



3.

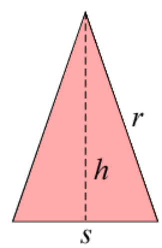


Hopefully your estimates for all three circles were slightly greater than 3. In fact, the exact answer is, as you may have already guessed, π. This relationship forms the very definition of what pi actually is. Before the symbol π was standard notation, the concept of pi was referred to as "the quantity which, when the diameter is multiplied by it, yields the circumference" in published works.

So, the true definition of pi is **the ratio of a circle's circumference to its diameter.** In mathematical notation this is written as $\pi = \frac{C}{d}$ which is equivalent to saying $\pi = \frac{C}{2r}$ where r is the radius. You may be more familiar with the rearranged version, $C = \pi d = 2\pi r$ which gives the circumference of a circle.

Notice that this is an astounding result. This tells us that for any circle, of any size, the ratio of its circumference to its diameter is *always the same.*

Naturally, since we know the circumference, we might also want to know the area of a circle. For this, let's revisit the concept of n-sided polygons inscribed in a unit circle. Each of the isosceles triangles looks like this:



Where r is the radius of the circle, h is the height of the triangles, and s is the side length of the polygon. The area of each triangle is $\frac{1}{2}sh$. As we saw before, there are n triangles in the polygon, so the area of the polygon is $\frac{n}{2}sh = \frac{h}{2}ns$.

Notice that ns is the number of sides of the polygon multiplied by the side length. Thus, ns is the perimeter of the polygon. As n increases, we get closer and closer to a circular shape. Thus we can replace ns with $2\pi r$. So we have that the area of the polygon as it approaches a circular shape is $\frac{h}{2}2\pi r = h\pi r$.

Finally, notice that as the polygon approaches a circular shape, the height of the triangles will approach the radius of the circle. (If you have trouble seeing this, refer back to the pictures on page 3.) Thus, we can replace h with r and we get that the area of the polygon as it approaches a circular shape is $A = r\pi r = \pi r^2$.

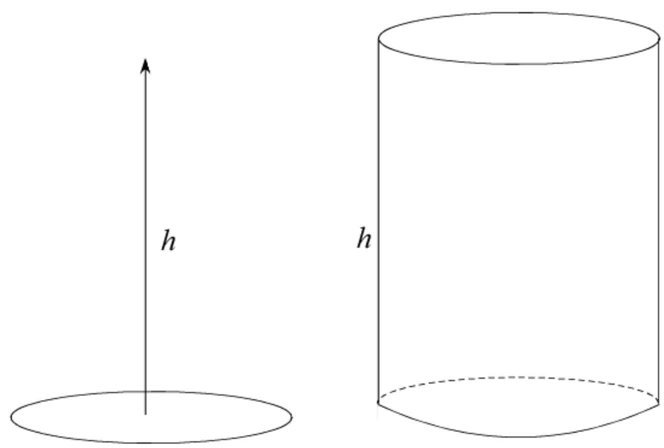
Try it out:

1. Calculate the area of a circle with radius 7 cm. 49π cm^2

2. Calculate the area of a circle with circumference 6π m. 9π m^2

WU 8 and 9 tested your knowledge of cylinders. Now that we know the area of a circle, we can extend this to volumes. We're going to take a look at cylinders in this lesson because they fall directly out of the concept of circular areas.

Imagine you place a circle on the ground. Then, you lift the circle to some height h above the ground. Imagine that as you do so, the circle is traced at every point in the air that it has passed through. You have just built a cylinder!



From this construction, you can logically see that the formula for the volume of the cylinder must be the area of the circle multiplied by the amount of times it was traced through space. In the case of a cylinder, that amount is the height h. Thus, the formula for the volume of a cylinder is $V = \pi r^2 h$.

Another quantity you should know how to find is the surface area of the cylinder. Well, if we were to "unfold" the cylinder, we would see it is made of two circles and a rectangle. The two circles are just the top and the bottom of the cylinder. The rectangle has "width" h and "length" equal to the circumference of the top (or bottom). Thus, the surface area of a cylinder is $SA = 2\pi r^2 + 2\pi rh$.

Try it out:

1. Find the volume of a cylinder with a base area of 5 cm^2 and height of 6 cm. 30 cm^3

2. Find the volume and surface area of a cylinder with a base diameter of 10 cm and height of 6 cm. $r = 5$ cm, $h = 6$ cm. $\therefore V = 150\pi$ cm^3, $SA = 110\pi$ cm^2.

Wrap-Up

Today you focused on a single fabulous number. You may not appreciate the greatness that is π now, but as you advance in your studies, you will continue to see just how useful pi can be.

Problem Set

Complete problems 1-14 without a calculator. You may use a calculator for problem 15. You may find the Pythagorean Theorem useful for some of the problems.

1. Redo the warm-up. See if you can answer more questions than before!

2. Pi is an irrational number. What does it mean to be an irrational number?
   An irrational number is a number that cannot be written as a ratio of two integers. It cannot be represented as a terminating or repeating decimal.

3. Calculate the area of a regular 174-gon using the isosceles triangle technique. 3.140910

4. What is the definition of pi? The ratio of a circle's circumference to its diameter.

5. What is the area of circle with diameter 16 cm? $r = 8$ cm $\therefore A = 64\pi$ cm^2.

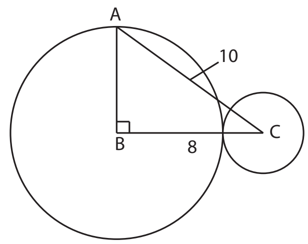
6. What is the circumference of a circle with radius 16 cm? 32π cm

7. What is the volume of a cylinder with base radius 2 mm and height 3 m?
   $r = 2$ mm, $h = 3000$ mm $\therefore V = 12\,000\pi$ mm^3.

8. What is the surface area of a cylinder with base diameter 4 mm and height 3 m?
   $r = 2$ mm, $h = 3000$ mm $\therefore SA = 12\,008\pi$ mm^2.

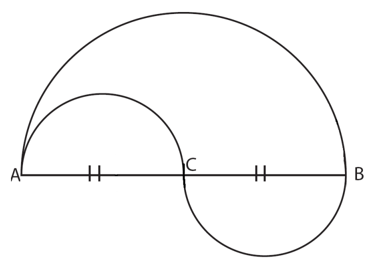
9. Thor's hammer is a compound shape. It is made of a cube with side length 7 cm and a cylinder with base circumference 6π cm and height 20 cm. What is the volume and surface area of Thor's hammer?
   $V_{cube} = 7^3 = 343$ cm^3 & $V_{cyl} = 3^2 \times 20 \times \pi = 180\pi$ cm^3 $\therefore V_{hammer} = 343 + 180\pi$ cm^3
   $SA_{cube} = 7^2 \times 6 = 294$ cm^2; $SA_{cyl} = (2 \times 3^2 \times \pi) + (6\pi \times 20) = 138\pi$ cm^2
   However, the circle where the cylinder meets the cube is not part of the surface area. It must be subtracted from the surface area of the cube and the cylinder.
   $\therefore SA_{hammer} = 294 + 138\pi - 2 \times 3^2\pi = 294 + 120\pi$ cm^2

10. In the diagram below, the line segment BC joins the centres of the two circles. If $BC = 8$ and $AC = 10$, then what is the circumference of the smaller circle?



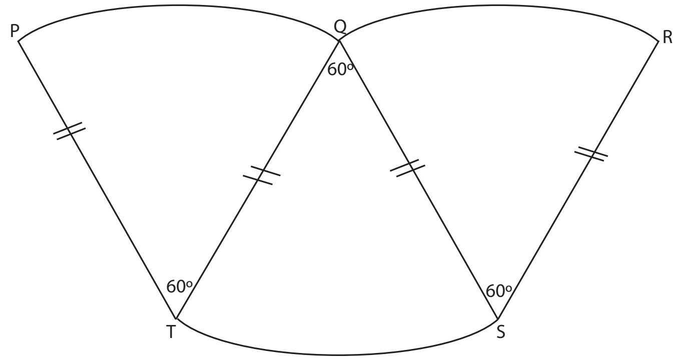
By Pyth Thm, the radius of the large circle AB is 6. So the radius of the small circle is $BC - 6 = 2$. Thus the circumference of the small circle is 4π.

11. In the diagram below, $AC = CB = 10$ m, where AC and CB are each the diameter of the small equal semi-circles. The diameter of the larger semi-circle is AB. In travelling from A to B, it is possible to take one of two paths. One path goes along the semi-circular arc from A to B. A second path goes along the semi-circular arc from A to C and then along the semi-circular arc from C to B. What is the difference in the lengths of these two paths?



The length of the path along semi-circle AB is $20\pi \div 2 = 10\pi$. The length of the path along the two smaller semi-circles is $10\pi \div 2 \times 2 = 10\pi$. $\therefore$ The difference in lengths of these two paths is 0.

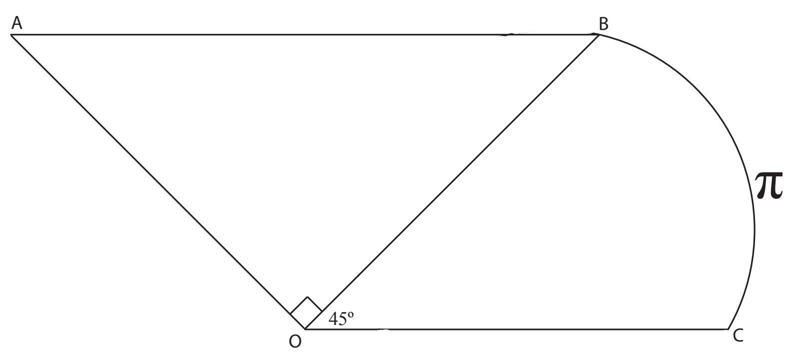
12. In the diagram below, circular arcs PQ, QR, and ST have centres S, T, and Q respectively. If PT equals one unit, then what is the perimeter of figure $PQRST$?



Notice that PT is the radius of a circle. Thus the full circle with radius 1 has circumference 2π. But each wedge is $60°$. Recalling that $360°$ makes a full circle, each wedge is one sixth of a full circle. Thus each circular arc is one sixth of the circumference of a circle with radius 1.

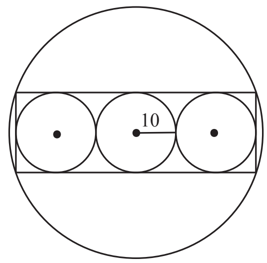
The perimeter of the figure is $PQ + QR + RS + ST + TP = \frac{\pi}{3} + \frac{\pi}{3} + 1 + \frac{\pi}{3} + 1 = 2 + \pi$

13. Determine the area of the figure below if the distance OB is twice OA, and BC is a portion of a circle's circumference.



$45°$ is one eighth of a full circle. Thus, BC is one eighth of a circle's circumference. This circle would have circumference 8π. This means the circle's radius OB and OC would be 4. Since OB is twice OA, OA is 2. Since it's a right triangle, we now see that the area of AOB is $(2 \times 4) \div 2 = 4$. Going back to the wedge, the full circle would have area 16π, so the wedge has area 2π. $\therefore$ the area of the figure is $4+2\pi$.

14. Three circles, each with radius of 10 cm, are drawn tangent to each other so that their centres are all in a straight line. These circles are inscribed in a rectangle which is inscribed in another circle. What is the area of the largest circle?



The width of the rectangle is 60 and its height is 20. By Pyth Thm, the diagonal of the rectangle is $\sqrt{4000} = 2\sqrt{1000}$. Notice the diagonal of the rectangle is the diameter of the large circle. Thus, the radius of the large circle is $2\sqrt{1000} \div 2 = \sqrt{1000}$. Then, the area of the large circle is $\sqrt{1000}^2\pi = 1000\pi$. Alternatively, you could look at the smaller triangle with legs 30 and 10.

15. π **in the Sky** - This question courtesy of NASA

The Curiosity Mars rover doesn't have an odometer like those found in cars, so rover drivers calculate how far the rover has driven based on wheel rotations. Since landing on Mars in August 2012, Curiosity's 50-centimeter-diameter wheels have rotated 3689.2 times in 568 sols (Martian days). How many kilometers has Curiosity traveled?

$C = \pi d = 50\pi$ cm ≈ 157.1 cm

Now multiply circumference by number of rotations to find distance traveled:

$157.1 \times 3689.2 = 579\,573.32$ cm

$\therefore$ Curiosity has travelled *approximately* 5.8 km.

Experiment

You've already seen Archimedes' n-gon approach to approximating pi. There is another interesting experiment, called Buffon's Needle, that can be used to approximate pi.

On the next page, you will see 5 lines spaced 40 mm apart (when printed out). Measure how far apart they are just in case. If they are not 40 mm apart, write down the actual spacing. You will need a match with the head cut off. Make sure the match is shorter than the line spacing (you may need to cut the match shorter). Write down the length of your match in mm.

You are going to drop the match from a height of about 5 cm onto the page 100 times. Make sure the paper is on a flat surface. Record each drop result with a tally mark in the table below. If the match rolls off the paper or does not fall on the area where the lines are, the drop doesn't count.

Touches or Crosses Line	**Does Not Touch Line**
Total:	Total:

Now that you've dropped the match 100 times, find the proportion, p, in decimal form of matches crossing the line. So for example, if 47 matches crossed the line, this means that the proportion of matches was $p = \frac{47}{100}$ since you dropped 100 matches. In decimal form this would be $p = 0.47$.

Finally, use Buffon's formula for the approximation of pi. L represents the length of your match, x is the line spacing, and p is the proportion in decimal form of matches crossing the line.

$$\pi \approx \frac{2L}{xp} = 2L \div xp$$

How good of an approximation was this? If you're interested, repeat the experiment with longer and shorter match lengths (always shorter than the line spacing) to see how this affects the result.

Extend Your Knowledge

If you've enjoyed learning about pi and would like to broaden your understanding of this wonderful number as well as challenge yourself, these sections are for you! While the following concepts are very interesting, they can be difficult to grasp.

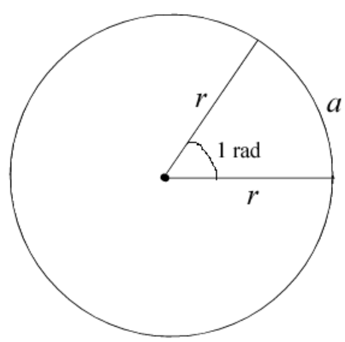
Warm-Up

1. How many degrees make a full circle? $360°$
2. How many radians make a full circle? 2π
3. What is the **exact** value of tau (τ)? 2π

Radians

WU 1 and 2 tested your understanding of radians. Radians are a standard unit of angular measure. While you may be used to using degrees, radians are a much more natural unit. Most of mathematics is actually done in radians. What's so beautiful about radians is that they are just numbers on the number line. Degrees are, well, degrees! You must convert them and you cannot really do very much with them!

A radian is technically defined as the angle formed when the arc length a of a circle is equal to the radius r.

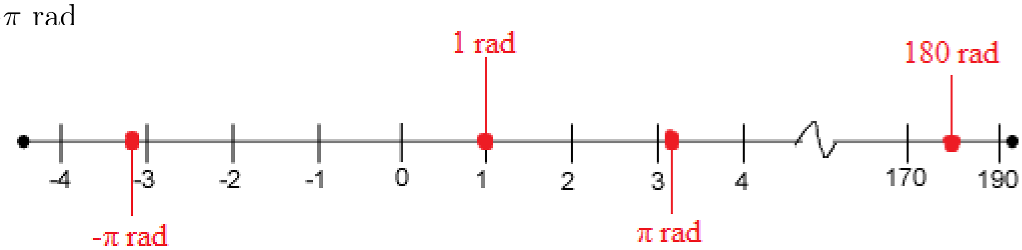


So, recalling your discovery with string and circles, if the diameter wraps around the circle π times, then the radius must wrap around the circle 2π times. Thus, 2π radians make a full circle. Then, since $360°$ make a circle, we have the conversion $2\pi = 360°$. Notice there are no units after 2π - this is because radians are *just a number*. Occasionally you may see "rad" after a radian measure, but this is normally only used in the learning stages.

Try it out:

Mark and label the following radian measures on the number line below:

- 1 rad
- π rad
- 180 rad
- $-\pi$ rad



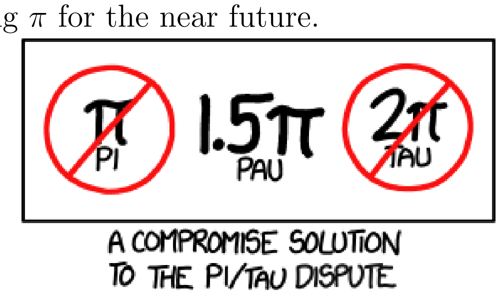
The Great Tau Debate

WU 3 tested your knowledge of tau. You've seen just how wonderful pi can be. But, in recent years, certain people in the mathematical world have insisted that "pi is wrong". What do they mean? Well, the concept and values of pi are not wrong. What they are saying is that the way pi has been defined is perhaps inelegant.

The suggested solution is pretty simple. Instead of using pi as our circle constant, use the value of 2π and call it tau (in symbolic form, τ). There are arguments for this new circle constant, such as the fact that it may make certain formulations easier. Perhaps the biggest push for τ has been the fact that it is visually easier. Recall that 2π (rad) makes a full circle. But using τ, a full circle would be τ (rad). A sixteenth of a circle would be $\frac{\tau}{16}$ instead of $\frac{\pi}{8}$. You can read the full *Tau Manifesto* online outlining the arguments for tau.

There are also many arguments against introducing tau and keeping pi. You can also read the full *Pi Manifesto* online outlining the arguments against tau.

While you can decide for yourself which you prefer, the world of mathematics will continue using the delicious-sounding π for the near future.



A great webcomic from XKCD: http://xkcd.com/1292/

Try it out:

Convert 45° to radians. What is this in terms of tau? $45° = \dfrac{\pi}{4}$ rad $= \dfrac{\tau}{8}$